Linear Algebra Review 21/5/22

① Multiplication of matrices

let $A = (a_{ij})$ be an ~~m×n~~ $l \times n$ matrix
let $B = (b_{ij})$ be an $n \times m$ matrix
] both over $\mathbb{R}$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$ where AB is an $l \times m$ matrix

② solving a system with non-unique soln:

~~x + y + z = 1~~

$$(A|b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\substack{r_2 = r_2 - r_1 \\ r_3 = r_3 - 3r_1}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1−3
5−3

$\therefore$ $x + z = 2$, $y = -1$

so set $z = \alpha$, $(x, y, z) = (2 - \alpha, -1, \alpha) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ✓

③ In row reduction, $c(i)$ is the position of the first non-zero entry on that row.
i is the row number.

④ solving equations $(A|b)$

- if we have a $\underline{0}\,|\,1 \rightarrow$ ✗ and no solutions.
- if ~~we~~ ~~sete~~ rewrite row in equation form, write out leading one in terms of free variables.

$$\left(\begin{array}{cccc|c} 1 & 0 & 9/5 & -1/5 & 4/5 \\ 0 & 1 & -2/5 & 3/5 & 3/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{let } \alpha = x_3,\ \beta = x_4$$

$\therefore$ $x_1 = \frac{4}{5} - \frac{9}{5}\alpha + \frac{1}{5}\beta$

$x_2 = \frac{3}{5} + \frac{2}{5}\alpha - \frac{3}{5}\beta$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -9/5 \\ 2/5 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1/5 \\ -3/5 \\ 0 \\ 1 \end{pmatrix}$$

row reduced form is unique!

Elementary Matrices

R1 $r_i = r_i + \lambda r_j$

$$\begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \iff r_1 = r_1 + \lambda r_3 \qquad i,j = (1,3)$$

R2 $r_i \longleftrightarrow r_j$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \iff r_2 \longleftrightarrow r_3$$

R3 $r_i = \lambda r_i$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \iff r_1 = \lambda r_1$$

~~elementary matrices~~ $\iff$

EROs $\iff$ matrix multiplication on left by EM.

Field Definition

$(K, +, \cdot)$ satisfying

A1 $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in K$

A2 $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \quad \forall \alpha, \beta, \gamma \in K$

A3 $\alpha + 0 = 0 + \alpha = \alpha \quad \forall \alpha \in K$

A4 $\forall \alpha \in K, \ \exists -\alpha \in K$ s.t $\alpha + (-\alpha) = -\alpha + \alpha = 0$

M1 $\alpha \cdot \beta = \beta \cdot \alpha \quad \forall \alpha, \beta \in K$.

M2 $(\alpha \cdot \beta)\gamma = \alpha \cdot (\beta \cdot \gamma) \quad \forall \alpha, \beta, \gamma \in K$

M3 $1 \cdot \alpha = \alpha \cdot 1 = \alpha \quad \forall \alpha \in K$

M4 $\forall \alpha \in K, \ \exists \alpha^{-1} \in K$ s.t. $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$

D $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \quad \forall \alpha, \beta, \gamma \in K$

Vector space Def

A vectorspace V over a field K is ~~a set 2V together~~

- A set V
- with an element $0 \in V$
- with two maps: ~~$\alpha \in K \to$~~ ~~[scalar~~
  - ⓐ scalar multiplication $\cdot : K \times V \to V$
  - ⓑ addition $+ : V \times V \to V$
- satisfying the following axioms $\forall \underline{u}, \underline{v}, \underline{w} \in V$ and $\alpha, \beta \in K$

(i) **A1** $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

A2 $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

A3 $\underline{0} + \underline{v} = \underline{v} + \underline{0} = \underline{v}$

A4 $\forall \underline{v} \in V \ \exists -\underline{v} \in V$ s.t. $\underline{v} + (-\underline{v}) = -\underline{v} + \underline{v} = \underline{0}$

(ii) $\alpha(\underline{u} + \underline{v}) = \alpha \underline{u} + \alpha \underline{v}$

(iii) $(\alpha + \beta)\underline{u} = \alpha \underline{u} + \beta \underline{u}$

(iv) $(\alpha\beta)\underline{u} = \alpha(\beta \underline{u})$

(v) $1 \cdot \underline{v} = \underline{v}$

Isomorphism
if T is linear map
and T is a bijection

Linear Transformations

Let U, V be two vector spaces over a field K. A linear map $T : U \to V$ is a function satisfying

$$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v}) \qquad \forall \underline{u}, \underline{v} \in U$$

$$T(\alpha \underline{u}) = \alpha T(\underline{u}) \qquad \forall \alpha \in K, \ \underline{u} \in U$$

or simply $T(\alpha \underline{u} + \beta \underline{v}) = \alpha T(\underline{u}) + \beta T(\underline{v}) \quad \forall \alpha, \beta \in K, \ \underline{u}, \underline{v} \in U$

$T(\underline{x}) = A\underline{x}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} = \begin{pmatrix} a_{11}(u_1 + v_1) + \cdots \\ \vdots \\ \cdots \end{pmatrix} = A\underline{u} + A\underline{v}$$

$T(\underline{u} + \underline{v}) = A(\underline{u} + \underline{v}) =$

$T(\alpha \underline{u}) =$

Def: $\underline{v}_1, \dots, \underline{v}_n \in V$ <u>linearly independent</u> if

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$$

$$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$$

Matrix check: leading one in every column.

Def: $\underline{v}_1, \dots, \underline{v}_n \in V$ <u>span</u> V if $\forall \underline{v} \in V$

$\exists \alpha_1, \dots, \alpha_n \in K$ s.t.

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{v}$$

matrix check: non zero rows.

Def: $\underline{v}_1, \dots, \underline{v}_n$ form basis of V if L.I & S.

every $\underline{v} \in V$ can be written <u>uniquely</u> as

$$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n .$$

matrix check: identity matrix.

The scalars $\alpha_1, \dots, \alpha_n$ are the <u>coordinates</u> of $\underline{v}$ wrt basis.

the linear map between the coordinates of $\underline{v}$ wrt. basis and the vector $\underline{v} \in V$ is a isomorphism.

<u>Properties perserved under isomorphisms</u>

1. linear independence
2. spanning
3. Forms a basis.

[all bases contain same number of vectors,
too many & lose linear independence,
too few & lose spanning.
n is just right!] $\Longrightarrow$ $\dim(V) = n$
[number of vectors in basis]

Extending linear independence

① pick a basis of V

② Row reduce $(u_1, u_2, \dots, u_n, v_1, v_2, v_i)$
[matrix formed from vectors given plus standard basis]

③ vectors corresponding to columns with leading ones are L.I.

Reducing spanning sets

~~Every~~ Any finite spanning set of V contains a basis

$\Rightarrow$ row reduce and pick ~~the~~ ~~zero rows~~ the identity matrix columns.

Pf: Assume minimality & assume linear dependence. Contradiction.

$\dim(V \times W) = \dim(V) + \dim(W)$.

Subspaces

A subspace of V, $W \subseteq V$ satisfying
(i) $\forall \underline{u}, \underline{v} \in W$, $\underline{u} + \underline{v} \in W$
(ii) $\forall \alpha \in \mathbb{F}, \underline{u} \in W$ $\alpha \underline{u} \in W$

linear maps uniquely determined by their actions on a basis

W_1 and W_2 are two subspaces of U

- $W_1 \cap W_2$ is also a subspace.
- $\dim(W_1) \leq \dim(V)$

- columns of A are the coordinates of the transformed old basis vectors wrt the new basis

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Linear Transformations and Matrices

$T: U \to V$ $\quad \dim(U) = n$, $\dim(V) = m$.

Basis of U: $\underline{e}_1, \dots, \underline{e}_n$
Basis of V: $f_1, \dots, f_m$

For $1 \leq j \leq n$, $T(\underline{e}_j) \in V$
so can write $T(\underline{e}_j)$ as a linear comb. of $f_1, \dots, f_m$

$$T(\underline{e}_1) = \alpha_{11} f_1 + \dots + \alpha_{m1} f_m$$
$$\vdots$$
$$T(\underline{e}_n) = \alpha_{1n} f_1 + \dots + \alpha_{mn} f_m$$

more compactly

$$\boxed{T(\underline{e}_j) = \sum_{i=1}^{m} \alpha_{ij} f_i}$$

coefficients α_{ij} form an ~~[illegible]~~ $m \times n$ matrix

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

A is the matrix of the linear map T wrt the chosen bases of U and V.

$\boxed{j\text{th column of } A \iff \text{coordinates of } T(\underline{e}_j) \text{ wrt. basis of } V}$

let $T: U \to V$ be a linear map. let $A = (a_{ij})$ represent T wrt the chosen bases of U and V. let $\underline{u}$ and $\underline{v}$ be the column vectors of ~~two~~ coordinates of to vectors $\underline{u} \in U$ and $\underline{v} \in V$ wrt same bases.

$$T(\underline{u}) = \underline{v} \iff A\underline{u} = \underline{v}$$

$$\underline{u} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{F}^n \quad \underline{v} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \in \mathbb{F}^m$$

Pf: $T(\underline{u}) = T\left(\sum_{j=1}^{n} \lambda_j \underline{e}_j\right)$ [$\underline{u} = \lambda_1 \underline{e}_1 + \dots + \lambda_n \underline{e}_n$ ← write $\underline{u}$ wrt basis of U]

$= \sum_{j=1}^{n} \lambda_j T(\underline{e}_j)$ [T is a linear map]

$= \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} a_{ij} \underline{f}_i \right)$ [Transformed basis vectors of U can be written wrt basis vectors of V (?)
$T(\underline{e}_j) = a_{1j}\underline{f}_1 + \dots + a_{mj}\underline{f}_j$]

$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \lambda_j \right) \underline{f}_i$ [interchanging sums]

$= \sum_{i=1}^{m} \mu_i \underline{f}_i$ [$\mu_i = \sum_{j=1}^{n} a_{ij}\lambda_j$ is the entry in the ith row of the column vector $A\underline{u}$]

$= \underline{v}$

Meaning

- choosing basis for $U \Rightarrow$ unique coordinates $\forall \underline{u} \in U$
- " " $V \Rightarrow$ " " $\forall \underline{v} \in V$
- Applying $T(\underline{u}) \iff$ column vector of coordinates of $\underline{u}$ getting multiplied by the matrix that represents T.
- irrespective of 'arbitrarily' chosen bases!

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matrix of:
- $T_1 + T_2$ is $A + B$
- λT is λA
- $T_1 T_2$ is AB

$$T: U \longrightarrow V$$

- $\mathrm{Im}(T) = \{ T(\underline{u}) : \underline{u} \in U \}$ [subspace of V]
- $\ker(T) = \{ \underline{u} \in U : T(\underline{u}) = \underline{0}_V \}$ [subspace of U]
- $\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$, $\mathrm{nullity}(T) = \dim(\ker(T))$
- $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(U)$ [pf: pick a basis of $\ker(T)$ and extend to a basis of U, the extended basis vectors will span $\mathrm{Im}(T)$].

If $\dim(U) = \dim(V) = n$, following are equivalent. [square matrices here]

- T is surjective
- $\mathrm{rank}(T) = n$
- $\ker(T) = \{0\}$
- $\mathrm{nullity}(T) = 0$
- T is injective
- T is bijective
- T is an isomorphism.

$\mathrm{Im}(T) = V$

rank-nullity

def

If $\ker(T) = \{0\}$, $T(\underline{u}_1) = T(\underline{u}_2)$
$\Rightarrow T(\underline{u}_1 - \underline{u}_2) = \underline{0}$
$\Rightarrow \underline{u}_1 - \underline{u}_2 \in \ker(T) = \{0\}$
$\Rightarrow \underline{u}_1 = \underline{u}_2 \Rightarrow T$ injective!

$\Downarrow$

Subspaces Relations

~~dim~~ Let V be a vector space, $U, W \subseteq V$ subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$$

The rank of a matrix

- If $\underline{e}_1, \dots, \underline{e}_n$ basis of U, $\mathrm{rank}(T) =$ ~~size of~~ the largest size of a linearly independent subset of $T(\underline{e}_1), \dots, T(\underline{e}_n)$.
- A is an $m \times n$ matrix over K.
  - row space $= \{ \alpha_1 \underline{r}_1 + \dots + \alpha_m \underline{r}_m \mid \alpha_i \in K \}$
  - column space $= \{ \alpha_1 \underline{c}_1 + \dots + \alpha_n \underline{c}_n \mid \alpha_i \in K \}$

  row/column rank is dimension of these spaces. (always the same)
- $\mathrm{rank}(A) = \#$ non zero rows after reducing A to upper echelon form.

Inverses

- Matrices that are not invertible are singular.
- For T to have a left & right inverse, it must be a bijection, so only one get mapped to 0_V once. Hence $\ker(T) = \{0\}$ and $\operatorname{Im}(T) = V$.
- $ABB^{-1}A^{-1} = AA^{-1} = I_n \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$
- If $n \times n$ matrix invertible $\Rightarrow \operatorname{rank}(A) = n$. $\operatorname{rref}(A) = I_n$.
- If A is not invertible, $\operatorname{rank}(A) < n$.
- An invertible matrix a product of elementary matrices.
- Unique soln to system of equations when $\det(A) \neq 0$.
- $A\underline{x} = \underline{b}$ has solutions iff $b \in \text{columnspace}(A)$
  $\Rightarrow$ solutions: $\underline{x}$ + nullspace(A)
- $\det(A) = \sum_{\phi \in S_n} \operatorname{sign}(\phi) a_{1\phi(1)} a_{2\phi(2)} \cdots a_{n\phi(n)}$. $\det(A^T) = \det(A)$.
  works with column operations too.
- $\det(I_n) = 1$. $r_i \leftrightarrow r_j = \det(A) = -\det(B)$, two equal rows $\Rightarrow \det(A) = 0$ ...
- If $A = (a_{ij})$ is upper triangular, $\det(A) = \prod_{i=1}^{n} a_{ii}$
  [so compute a big determinant, row reduce & keep track of the sign].
- $n \times n$ matrix. $\det(A) = 0 \iff A$ is singular. [not invertible / T not bijection]
  $\operatorname{rank}(A) < n$.
- $\det(AB) = \det(A)\det(B)$.
- $M_{ij} = (i,j)$ th minor of A is determinant of matrix formed by deleting ith row & jth column.
- $C_{ij} = (-1)^{i+j} M_{ij}$ [$i+j$ even $\Rightarrow +$, $i+j$ odd $\Rightarrow -$].
- Get expansion along row/column.
- Adjugate matrix $\operatorname{adj}(A)$ = transpose of the matrix of cofactors.
- $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$. $A \operatorname{adj}(A) = \det(A) I_n = \operatorname{adj}(A) A$.

Eigenvalues & Eigenvectors

Informally:

eigenvectors
$\Downarrow$
direction in which linear map stretches/compresses

eigenvalues
$\Downarrow$
give the stretching factor

$T(\underline{v}) = \lambda \underline{v}$ [still travelling in same direction]

$A\underline{v} = \lambda \underline{v}$

$\det(A - xI) = 0$
$\downarrow$
eigenvalues depend on the field $\mathbb{F}$!

If A is upper triangular eigenvalues are diagonal entries

$A\underline{v} = \lambda v$

$\Rightarrow (A - \lambda I)\underline{v} = \underline{0}$

$\underline{v} \in$ nullspace$(A - \lambda I)$ — not invertible as only 0, $T(0) = 0v$

$\underline{v} \in \ker(A - \lambda I)$, $\underline{v} \neq \underline{0}$

$\Rightarrow A - \lambda I$ singular

$\Rightarrow \det(A - \lambda I) = 0$

Let $T: V \to V$ be a linear map. T is diagonal wrt some basis of V iff V has a basis consisting of eigenvectors of T

distinct eigenvalues
$\Downarrow$
linearly independent eigenvectors
$\Downarrow$
n distinct eigenvalues
$\Downarrow$
T diagonalisable.

Change of ~~Basis~~ & Equivalent matrices

$$\begin{array}{ccc} T: & U & \longrightarrow & V \\ & e_1, e_2, \dots, e_n & \xrightarrow{A} & f_1, f_2, \dots, f_m \\ & \downarrow P & & \downarrow Q \\ & e_1', e_2', \dots, e_n' & \xrightarrow[B]{} & f_1', f_2', \dots, f_m' \end{array}$$

$B = \cancel{P^{-1}AQ}\ QAP^{-1}$

[coordinate change from new basis to old basis is P]

- Two $m \times n$ matrices A and B are $\boxed{\text{equivalent}}$ if there exist invertible P and Q with $B = QAP$. [Q $m \times m$] [P $n \times n$]
  $\Leftrightarrow$
- A and B represent same linear map wrt different bases
- A ~~and~~ and B have the same rank.
- B obtained from A by EROs
- $E_s = \left(\begin{array}{c|c} I_{s \times s} & 0_{s, n-s} \\ \hline 0_{m-s,s} & 0_{m-s, n-s} \end{array}\right)$ is canonical form of $m \times n$ matrices under equivalence.

Similar Matrices

$$\begin{array}{ccc} T: & V & \longrightarrow & V \\ & e_1, \dots, e_n & \xrightarrow{A} & e_1, \dots, e_n \\ & P \uparrow & & \uparrow P \\ & e_1', \dots, e_n' & \xrightarrow{B} & e_1', \dots, e_n' \end{array}$$

$B = P^{-1}AP$

- Two $n \times n$ matrices over $\mathbb{F}$ are $\boxed{\text{similar}}$ if there exist an $n \times n$ invertible matrix P with $B = P^{-1}AP$.
- represent the same linear map wrt different bases.
- Similar matrices have same characteristic polynomial $\Rightarrow$ same eigenvalues.
- If A is a real symmetric matrix, then $\exists$ real orthogonal cob matrix w/ $P^{-1}AP = P^TAP$ diagonal.
- basis orthonormal if $\underline{b}_i \cdot \underline{b}_i = 1$, $\underline{b}_i \cdot \underline{b}_j = 0$
- A $\boxed{\text{symmetric}}$ if $A = A^T \to$ real eigenvalues.
- A orthogonal if $AA^T = A^TA = I_n \Leftrightarrow A^T = A^{-1}$.
  rows & columns form an orthonormal basis.
  $\to$ eigenvectors are orthonormal!