~~Chap~~ End of Linear Algebra MA10G

2/5/22

Def: $\boxed{\text{Diagonalisable:}}$ [two versions - linear maps & matrices]

A linear map $T: V \to V$ is diagonalisable if there is a basis for V such that the matrix A for T ~~mat~~ wrt this basis is diagonal. [$a_{ij} = 0$ if $i \neq j$]

$n \times n$ matrix A ~~is~~ is diagonalisable if there is a ~~linear map~~ basis for K^n such that the linear map ~~given by~~ $T: K^n \to K^n$ given by $T(\underline{x}) = A\underline{x}$ is diagonal.

Relationship between diagonal matrices & eigenvectors

Let $T: V \to V$ be a linear map. The matrix for T is diagonal wrt some basis of V $\iff$ V has a basis consisting of eigenvectors of T	Let A be an $n \times n$ matrix over K. Then A diagonalisable $\iff$ K^n has a basis of eigenvectors of A

Proof: [two statements equivalent from correspondence between linear maps and matrices and the definitions of eigenvalues & eigenvectors]

'$\Rightarrow$' Suppose the matrix $A = (a_{ij})$ of T is diagonal wrt the basis $\underline{e}_1, \dots, \underline{e}_n$ of V. We know the image of the ith basis vector is represented by the ith column of A; but since A is diagonal, this column has the single non-zero entry a_{ii}. So

$T(\underline{e}_i) = a_{ii}\underline{e}_i \iff A\underline{e}_i = a_{ii}\underline{e}_i \iff$ each basis vector $\underline{e}_i$ is an eigenvector of ~~A~~ A.

'$\Leftarrow$' Suppose that $\underline{e}_1, \dots, \underline{e}_n$ is a basis of V consisting entirely of eigenvectors of T. Then for each i we have $T(\underline{e}_i) = \lambda_i \underline{e}_i$ for some $\lambda_i \in K$. [But as the image of the ith basis vector is the ith column of the matrix A for T wrt that basis.] So the matrix A wrt this ~~bas~~ basis is the diagonal matrix $A = (a_{ij})$ with $a_{ii} = \lambda_i \; \forall i$. $\square$

$\boxed{\text{Can't diagonalise every matrix}}$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \det(A - xI) = \left|\begin{matrix} 1-x & 1 \\ 0 & 1-x \end{matrix}\right| = 0 \Rightarrow (1-x)^2 = 0$

So $\lambda = 1$. $\ker(A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

can't diagonalise matrix A

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\text{span}(\ker(A - I)) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so ~~we~~ can't find two linearly independent eigenvectors. So no basis of K^2 of eigenvectors distinct eigenvalues $\Longrightarrow$ corresponding eigenvectors linearly independent

thm
~~Proof~~: lets use ~~induction on r~~. ~~Suppose~~ Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of T, let $\underline{v}_1, \dots, \underline{v}_r$ be corresponding eigenvectors. Then $\underline{v}_1, \dots, \underline{v}_r$ are linearly independent.

Proof: $\boxed{\text{Idea is induction on } r}$

- $r=1$. λ_1 has eigenvector $\underline{v}_1$ which is clearly linearly independent as $\underline{v}_1 \neq \underline{0}$.
- Now for $r > 1$ suppose $\exists\, \alpha_i \in \mathbb{R}$ s.t.

$$\alpha_1 \underline{v}_1 + \dots + \alpha_r \underline{v}_r = \underline{0} \qquad (*)$$

Applying T yields

$$\alpha_1 T(\underline{v}_1) + \dots + \alpha_r T(\underline{v}_r) = \underline{0}$$

and as $\underline{v}_1, \dots, \underline{v}_r$ are eigenvectors, we have

$$\alpha_1 \lambda_1 \underline{v}_1 + \dots + \alpha_r \lambda_r \underline{v}_r = \underline{0} \qquad (**)$$

we have two pieces of information about the $\{\alpha_i \underline{v}_i\}$ so lets combine.
Taking $(**) - \lambda_1 (*)$ yields

$$\alpha_2(\lambda_2 - \lambda_1)\underline{v}_2 + \dots + \alpha_r(\lambda_r - \lambda_1)\underline{v}_r = \underline{0}$$

[as we have 1 less & prove for 1 more]
[$r-1$ assumption

By the induction hypothesis, the $\underline{v}_2, \dots, \underline{v}_r$ are linearly independent so $\alpha_i(\lambda_i - \lambda_1) = 0$ $\forall i$. we know $\lambda_i \neq \lambda_1$ $\forall i$ as the λ_i are distinct so $\alpha_i = 0$

$\Longrightarrow$ $\underline{v}_1, \dots, \underline{v}_r$ linearly independent. $\square$

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linear map has n distinct eigenvalues $\dim(V) = n$ $\Longrightarrow$ T (or A diagonalisable)

Pf: $\underline{v}_1, \dots, \underline{v}_n$ linearly independent by above and $\dim(V) = n$ so $\underline{v}_1, \dots, \underline{v}_n$ must form a basis of V so we have a basis of eigenvectors

$\Longrightarrow$ T diagonalisable.

NOTE: can have ~~less~~ n non-distinct eigenvalues but matrix could still be diagonalisable if we end up with distinct eigenvectors that span ~~the~~ V

Change of basis & Equivalent matrices

2/5/22 — chapter 14

Let U be a vector space $\dim(U) = n$.

$\underline{e}_1, \dots, \underline{e}_n$ — 1 basis of U

$\underline{e}_1', \dots, \underline{e}_n'$ — 2nd basis of U

The matrix P of the identity map $I_U : U \to U$ using basis $\underline{e}_1, \dots, \underline{e}_n$ in domain and using basis $\underline{e}_1', \dots, \underline{e}_n'$ in the range of I_U

P is the change of basis matrix from the ~~matrix~~ basis of $\underline{e}_i$'s to the basis $\underline{e}_i'$s

Take $P = (\sigma_{ij})$ [matrix]. As I_U takes a vector written in $\underline{e}_1, \dots, \underline{e}_n$ to a vector written as a linear combination of $\underline{e}_1', \dots, \underline{e}_n'$, we can say

$$\boxed{I_U(\underline{e}_j) = \underline{e}_j = \sum_{i=1}^{n} \sigma_{ij} \underline{e}_i'} \quad \text{for } 1 \leq j \leq n \qquad (*)$$

so the columns of the change of basis matrix of P are the coordinates of the old basis vectors $\underline{e}_i$ wrt the new basis vectors $\underline{e}_i'$s.

The change of base matrix is invertible

e.g. If P is the change of base matrix from the basis $\underline{e}_1, \dots, \underline{e}_n$ to $\underline{e}_1', \dots, \underline{e}_n'$ then and Q is the change of base matrix from the basis of $\underline{e}_i'$s to the basis of $\underline{e}_i$s, then $P = Q^{-1}$

Proof: Take the composition of linear maps

$$I_U : U \xrightarrow[Q]{I_U} U \xrightarrow[P]{I_U} U$$

basis for U: $\underline{e}_i'$ — $\underline{e}_i$ — $\underline{e}_i'$

I_n

~~overall~~

[identity matrix ~~as~~ overall as same ~~domain~~ basis for domain and range]

$\implies$ so $I_n = PQ \iff I_n = QP \iff P = Q^{-1}$

$\square$

the change of base matrix turns vectors coordinates wrt **old** basis into the same vectors coordinates wrt the **new** basis

E.g.

~~Ex~~ Let $\underline{v} \in U$. Let $\underline{v}_i$ be column vector of $\underline{v}$ wrt basis $\underline{e}_1, \dots, \underline{e}_n$

Let $\underline{v}_i'$ be column vector of $\underline{v}$ wrt basis $\underline{e}_1', \dots, \underline{e}_n'$

$$\Rightarrow \quad P\underline{v}_i = \underline{v}_i'$$

Effect of change of bases on Linear Maps

- Let $T: U \to V$ be a linear map. $\dim(U) = n$, $\dim(V) = m$.

U: basis $\underline{e}_1, \dots, \underline{e}_n$; basis $\underline{e}_1', \dots, \underline{e}_n'$

V: basis $\underline{f}_1, \dots, \underline{f}_m$; basis $\underline{f}_1', \dots, \underline{f}_m'$

- Let $A = (a_{ij})$ be the $m \times n$ matrix of T wrt bases $\{\underline{e}_i\}$ and $\{\underline{f}_i\}$ of U and V.

so $$T(\underline{e}_j) = \sum_{i=1}^{m} a_{ij} \underline{f}_j \quad \text{for } 1 \le j \le n$$

- Let $B = (b_{ij})$ be the $m \times n$ of T wrt. bases $\{\underline{e}_i'\}$ and $\{\underline{f}_i'\}$ of U and V.

so $$T(\underline{e}_j') = \sum_{i=1}^{m} b_{ij} \underline{f}_j' \quad \text{for } 1 \le j \le n$$

- Let the $n \times n$ matrix $P = (\sigma_{ij})$ be the change of ~~base~~ basis matrix from $\{\underline{e}_i\}$ to $\{\underline{e}_i'\}$

- Let the $m \times m$ matrix $Q = \{\tau_{ij}\}$ be the change of basis matrix from $\{\underline{f}_i\}$ to $\{\underline{f}_i'\}$

Aim: Find relationship between A and B wrt the change of bases matrices P and Q

$$\begin{array}{ccc} U & & V \\ e_1, \dots, e_n & \xrightarrow{A} & f_1, \dots, f_m \\ P \downarrow & & Q \downarrow \\ e_1', \dots, e_n' & \xrightarrow{B} & f_1', \dots, f_m' \end{array}$$

$$B = QAP^{-1}$$

rough sketch of ~~chasse~~ relationship between change of bases matrices and matrices for linear transformations w.r.t different bases

$$\begin{tikzcd}
U \arrow[rr, bend left, "T: U \to V"] & & V \\
\underline{e}_1, \dots, \underline{e}_n \arrow[rr, "A"] \arrow[d, "P"'] & & \underline{f}_1, \dots, \underline{f}_m \arrow[d, "Q"] \\
\underline{e}_1', \dots, \underline{e}_n' \arrow[rr, "B"] & & \underline{f}_1', \dots, \underline{f}_m'
\end{tikzcd}$$

Theorem: $BP = QA \iff B = QAP^{-1}$

two $m \times n$ matrices A and B represent the same linear map from an n-dimensional vector space to an m-d vector space

$\iff$

there exist invertible ~~matrices~~ $n \times n$ and $m \times m$ matrices P and Q with $B = QAP$

Proof: 1 $I_U : U \xrightarrow{P} U$

$\underline{e}_1, \dots, \underline{e}_n \qquad \underline{e}_1', \dots, \underline{e}_n'$

$T : U \xrightarrow{B} V$

$\underline{e}_1', \dots, \underline{e}_n' \qquad \underline{f}_1', \dots, \underline{f}_m'$

$\therefore \quad T \circ I_U : U \xrightarrow{BP} V$

$\underline{e}_1, \dots, \underline{e}_n \qquad \underline{f}_1', \dots, \underline{f}_m'$

2 $T : U \xrightarrow{A} V \qquad I_V : V \xrightarrow{Q} V$

$\underline{e}_1, \dots, \underline{e}_n \quad \underline{f}_1, \dots, \underline{f}_m \qquad \underline{f}_1, \dots, \underline{f}_m \quad \underline{f}_1', \dots, \underline{f}_m'$

$\therefore \quad I_V \circ T : U \xrightarrow{QA} V$

$\underline{e}_1, \dots, \underline{e}_n \qquad \underline{f}_1', \dots, \underline{f}_m'$

so combining 1 and 2

$\implies BP = QA$

Equivalent matrices

Def: Two $m \times n$ matrices A and B are equivalent if there exist invertible P and Q with $B = QAP$

equivalent conditions:

1. A and B are equivalent
2. A and B represent same linear map wrt different bases
3. A and B have the $\boxed{\text{same rank}}$ — makes it easy to check if two matrices are equivalent.
4. B can be obtained from A by application EROs & ECOs ?

Proof: ① $\iff$ ② $\because$ A & B represent same linear map T if $\exists$ change of basis matrices P and Q s.t. $B = QAP^{-1}$ and we know that change of basis matrices are invertible so replace P with P^{-1} so there are invertible matrices with $B = QAP \iff$ A and B are equivalent.

② $\implies$ ③ Since A and B both rep. same linear map T, $\dim(Im(A)) = \dim(Im(B)) = \dim(Im(T))$ otherwise ✗ so $rank(A) = rank(B)$.

③ $\implies$ ④ $\because$ as A and B both have rank s, both can be brought into $m \times n$ block form

$$E_s = \left(\begin{array}{c|c} I_s & O_{s,n-s} \\ \hline O_{m-s,s} & O_{m-s,n-s} \end{array}\right)$$

by EROs and ECOs. As EROs & ECOs invertible, $A \xrightarrow{\text{operations}} E_s \xrightarrow{\text{operations}} B$

④ $\implies$ ① can write these ~~transformations~~ operations as a product of elementary row & column operations so $\exists$ elementary row matrices $R_1, \dots, R_r$ and EC matrices $C_1, \dots, C_s$ s.t. $B = R_r \dots R_1 A C_1 \dots C_s$ so take the product & it is invertible so $\exists$ invertible P and Q s.t. $B = QAP$. $\square$

note: any $m \times n$ matrix A is equivalent to the matrix E_s above where $s = rank(A)$. The form E_s is a canonical form for $m \times n$ matrices under equivalence.

Similar $\implies$ equivalent

equivalent $\not\Rightarrow$ Similar

Similar Matrices

V vector space over field k. $\mathrm{Dim}(V) = n$. $T: V \to V$ is the linear map

$$\begin{tikzcd} T: & V \arrow[rr, bend left] & & V \\ & \underline{e}_1, \dots, \underline{e}_n \arrow[rr, "A"] & & \underline{e}_1, \dots, \underline{e}_n \\ & \underline{e}_1', \dots, \underline{e}_n' \arrow[u, "P"] \arrow[rr, "B"] & & \underline{e}_1', \dots, \underline{e}_n' \arrow[u, "P"'] \end{tikzcd}$$

Setup:

- $\underline{e}_1, \dots, \underline{e}_n$ and $\underline{e}_1', \dots, \underline{e}_n'$ are two bases of V.
- A ~~is the~~ $= (a_{ij})$ is the matrix for T wrt $\{\underline{e}_i\}$
- $B = (b_{ij})$ is the matrix for T wrt $\{\underline{e}_i'\}$
- $P = (\sigma_{ij})$ is the change of base matrix from $\{\underline{e}_i'\}$ to $\{\underline{e}_i\}$

Opposite to before !!!

Thm: clearly, $B = P^{-1} A P$

Def: two $n \times n$ matrices over k are *similar* if there exists an $n \times n$ invertible matrix P with $B = P^{-1} A P$

In other words: two matrices are similar $\iff$ they represent the same linear map $T: V \to V$ w.r.t. ~~two~~ different bases of V

E.g. $T: \mathbb{R}^3 \to \mathbb{R}^3$ where
$$T(\underline{x}) = A_2 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \underline{x}$$
in basis (of eigenvectors)
$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
for $\mathbb{R}^3$ is
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
and computing change of basis matrix from new to old
$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
and computing
$$P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$
$$P^{-1} A_2 P = B \quad \text{~~as required~~}$$
so A_2 & B similar.

Similar matrices have the same characteristic polynomial & hence the same eigenvalues

Proof: Let A and B be similar matrices. Then $\exists P$ [invertible] with $B = P^{-1} A P$ so

$$\begin{aligned} \det(B - xI_n) &= \det(P^{-1} A P - xI_n) \\ &= \det(P^{-1}(A - xI_n)P) \\ &= \det(P^{-1}) \det(A - xI_n) \det(P) \\ &= \frac{\det P}{\det(P)} \det(A - xI_n) \\ &= \det(A - xI_n) \end{aligned}$$

so A and B have the same characteristic polynomial

$\implies$ same eigenvectors

Note: the different matrices corresponding to a linear map T are all similar so they all have the same characteristic equation so can refer to it as the characteristic polynomial of T.

$\Downarrow$

hard to find a E_s like row and column reduced canonical form like we do for equivalent matrices.

Similar vs Equivalent matrices

$T: U \longrightarrow V$	$T: V \longrightarrow V$
Two $m \times n$ matrices are **equivalent** if $\exists$ invertible P and Q with $B = QAP$	Two $n \times n$ matrices over $\mathbb{F}$ are **similar** if $\exists$ $n \times n$ invertible matrix P with $B = P^{-1}AP$

A & B have same characteristic polynomial so same eigenvalues.

Def: $\underline{v} = (a_1, \dots, a_n)$, $\underline{w} = (b_1, \dots, b_n)$, $\underline{v}, \underline{w} \in \mathbb{R}^n$.

$$\underline{v} \cdot \underline{w} = \sum_{i=1}^{n} a_i b_i \quad \text{[scalar product]}$$

orthonormal basis

A basis $\underline{b}_1, \dots, \underline{b}_n$ of $\mathbb{R}^n$ is **orthonormal** if

① $\underline{b}_i \cdot \underline{b}_i = 1$ for $1 \le i \le n$

② $\underline{b}_i \cdot \underline{b}_j = 0$ for $1 \le i, j \le n$ and $i \ne j$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\uparrow$ standard basis for $\mathbb{R}^3$ is orthonormal

Symmetric matrices

$n \times n$ matrix A symmetric if

$$A^T = A$$

orthogonal matrices

$n \times n$ matrix A is orthogonal if

$$A^T = A^{-1}$$

$$\iff AA^T = A^TA = I_n$$

omit pf:

A orthonormal matrix $A \in M_{2\times 2}(\mathbb{R})$ $\iff$ rows $\underline{r}_1, \dots, \underline{r}_n$ of A form an orthonormal basis of $\mathbb{R}^n$ $\iff$ columns $\underline{c}_1, \dots, \underline{c}_n$ of A form an orthonormal basis $\mathbb{R}^{n \cdot 1}$

A ~~@~~ is a real symmetric matrix

$\Downarrow$

A has only real eigenvalues

Pf: $\det(A - xI_n)$ has a root $\lambda \in \mathbb{C}$ which is an eigenvalue of A

$Av = \lambda v$ and transpose it $v^TA = \lambda v^T$ (3)

$A\bar{v} = \bar{\lambda}\bar{v}$ (4) ~~and transpose it~~

multiply (3) by $\bar{v}$ on the right.
multiply (4) by v^T on the left] compare: get $\lambda = \bar{\lambda}$

A is a real symmetric matrix. $\lambda_1 \lambda_2$ are two distinct eigenvalues of A [corresponding $\underline{v}_1, \underline{v}_2$]

$\Downarrow$

$\underline{v}_1 \cdot \underline{v}_2 = 0$

Let A be a real symmetric matrix $(n \times n)$. Then there exists a real **orthogonal matrix** P with $\underbrace{P^{-1}AP (= P^TAP)}_{\text{diagonal.}}$

$\downarrow$ change of basis matrix of normalised eigenvectors.

Pf: get an orthonormal basis of ~~$\mathbb{R}$~~ [eigenvectors]

so $P^{-1}AP$ diagonal Entries $(P^{-1}AP)_{ii} = \lambda_i$