

Def: Diagonalisable: [two versions - linear maps & matrices]

A linear map $T: V \rightarrow V$ is diagonalisable if there is a basis for V such that the matrix A for T wrt this basis is diagonal. [$a_{ij} = 0$ if $i \neq j$]

An $n \times n$ matrix A is diagonalisable if there is a ~~linear map~~ basis for k^n such that the linear map ~~given by~~ $T: k^n \rightarrow k^n$ given by $T(x) = Ax$ is diagonal.

Relationship between diagonal matrices & eigenvectors

Let $T: V \rightarrow V$ be a linear map. the matrix for T is diagonal wrt some basis of V

$\Leftrightarrow$

V has a basis consisting of eigenvectors of T

Let A be an $n \times n$ matrix over k . Then A diagonalisable

$\Leftrightarrow$

k^n has a basis of eigenvectors of A

Proof: [two statements equivalent from correspondence between linear maps and matrices and the definitions of eigenvalues & eigenvectors].

$\Rightarrow$ Suppose the matrix $A = (a_{ij})$ of T is diagonal wrt the basis $e_1, \dots, e_n$ of V . We know the image of the i th basis vector is represented by the i th column of A , but since A is diagonal, this column has the single non-zero entry a_{ii} . So

$T(e_i) = a_{ii}e_i \Leftrightarrow Ae_i = a_{ii}e_i \Leftrightarrow$ each basis vector e_i is an eigenvector of A .

$\Leftarrow$ Suppose that $e_1, \dots, e_n$ is a basis of V consisting entirely of eigenvectors of T . Then for each i we have $T(e_i) = \lambda_i e_i$ for some $\lambda_i \in k$.

[But as the image of the i th basis vector is the i th column of the matrix A for T wrt that basis.] So the matrix A wrt this basis is the diagonal matrix $A = (a_{ij})$ with $a_{ii} = \lambda_i \forall i$.

Can't diagonalise every matrix

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\det(A - xI) = \begin{vmatrix} 1-x & 1 \\ 0 & 1-x \end{vmatrix} = 0 \Rightarrow (1-x)^2 = 0$
 so $\lambda = 1$. $\ker(A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ can't diagonalise matrix A

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\text{span}(\ker(A - I)) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$
 so can't find two linearly independent eigenvectors. so no basis of k^2 of eigenvectors

distinct eigenvalues $\Rightarrow$ corresponding eigenvectors linearly independent

thm
~~Proof~~ Let's use ~~induction on~~. Suppose let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of T , let $v_1, \dots, v_r$ be corresponding eigenvectors, then $v_1, \dots, v_r$ are linearly independent.

Proof: Idea is induction on r .

• $r=1$. λ_1 has eigenvector v_1 which is clearly linearly independent as $v_1 \neq 0$.

• Now for $r > 1$ ~~we~~ suppose $\exists a_i \in \mathbb{K}$ s.t.

$$a_1 v_1 + \dots + a_r v_r = 0 \quad (*)$$

Applying T yields

$$a_1 T(v_1) + \dots + a_r T(v_r) = 0$$

and as $v_1, \dots, v_r$ are eigenvectors, we have

$$a_1 \lambda_1 v_1 + \dots + a_r \lambda_r v_r = 0 \quad (**)$$

we have two pieces of information about the $\{a_i v_i\}$ so let's combine.

Taking $(**) - \lambda_1 (*)$ yields

$$a_2 (\lambda_2 - \lambda_1) v_2 + \dots + a_r (\lambda_r - \lambda_1) v_r = 0$$

as we have 1 less a_i
prove for 1 more

[$r-1$ assumption]

By the induction hypothesis, the $v_2, \dots, v_r$ are linearly independent so $a_i (\lambda_i - \lambda_1) = 0$. we know $\lambda_i \neq \lambda_1$ as the λ_i are distinct so $a_i = 0$

$\Rightarrow v_1, \dots, v_r$ linearly independent. $\square$

linear map has n
distinct eigenvalues
 $\dim(V) = n$

$\Rightarrow$

T (or A) diagonalisable

Pr: $v_1, \dots, v_n$ linearly independent by above and $\dim(V) = n$
so $v_1, \dots, v_n$ must form a basis of V so

we have a basis of eigenvectors

$\Rightarrow T$ diagonalisable.

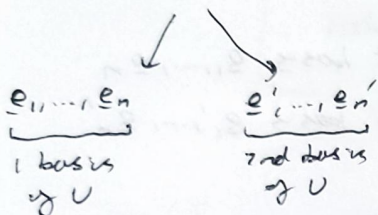
note: can have ~~be~~ n non-distinct eigenvalues but matrix could still be diagonalisable if we can up with n distinct eigenvectors.
that span V

Change of basis & Equivalent

21/5/22 monsoor Les

chapter 14

Let U be a vector space $\dim(U) = n$.



The matrix P of the identity map $I_U: U \rightarrow U$ using basis $e_1, \dots, e_n$ in domain and using basis $e'_1, \dots, e'_n$ in the range of I_U .

P is the change of basis matrix from the ~~matrix~~ basis e_i 's to the basis e'_i 's

Take $P = (\sigma_{ij})$ [matrix]. As I_U takes a vector written in $e_1, \dots, e_n$ to a vector written ~~a linear combination of~~ $e'_1, \dots, e'_n$, we can say

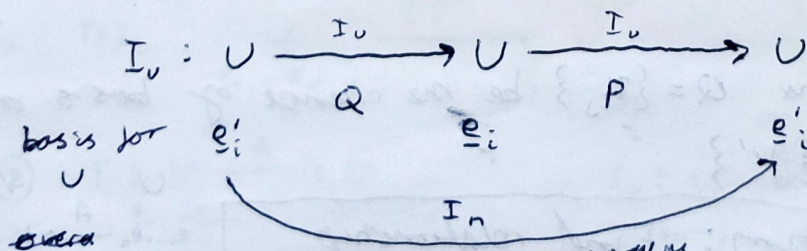
$$I_U(e_j) = e_j = \sum_{i=1}^n \sigma_{ij} e'_i \quad \text{for } 1 \leq j \leq n \quad (*)$$

so the columns of the change of basis matrix of P are the coordinates of the old basis vectors e_i wrt the new basis vectors e'_i 's.

The change of base matrix is invertible

e.g. If P is the change of base matrix from the basis $e_1, \dots, e_n$ to $e'_1, \dots, e'_n$ then and Q is the change of base matrix from the e'_i 's to the basis of e_i 's, then $P = Q^{-1}$

Proof: Take the composition of linear maps



[identity matrix as overall as same domain basis for domain and range]

$$\Rightarrow \text{so } I_n = PQ \Leftrightarrow I_n = QP \Leftrightarrow P = Q^{-1}$$

the change of base matrix
turns vectors coordinates wrt old basis
into the same vectors coordinates
wrt the new basis

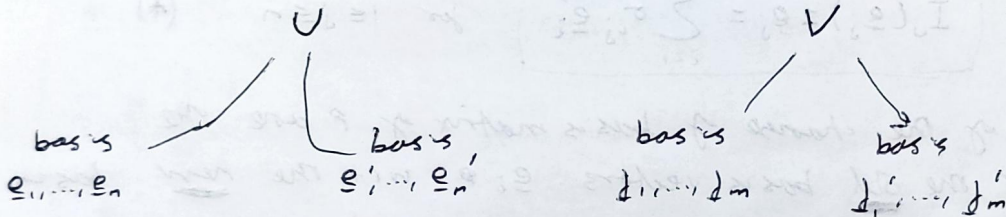
E.g.

Let $v \in U$. Let v_i be column vector of v wrt basis $e_1, \dots, e_n$
Let v'_i be column vector of v wrt basis $e'_1, \dots, e'_n$

$$\Rightarrow P v_i = v'_i$$

Effect of change of basis on Linear Maps

Let $T: U \rightarrow V$ be a linear map. $\dim(U) = n$, $\dim(V) = m$.



Let $A = (a_{ij})$ be the $m \times n$ matrix of T wrt bases $\{e_i\}$ and $\{d_i\}$ of U and V .

$$\text{so } T(e_j) = \sum_{i=1}^m a_{ij} d_i \quad \text{for } 1 \leq j \leq n$$

Let $B = (b_{ij})$ be the $m \times n$ of T wrt. bases $\{e'_i\}$ and $\{d'_i\}$ of U and V .

$$\text{so } T(e'_j) = \sum_{i=1}^m b_{ij} d'_i \quad \text{for } 1 \leq j \leq n$$

Let the $n \times n$ matrix $P = (p_{ij})$ be the change of ^{basis} ~~base~~ matrix from $\{e_i\}$ to $\{e'_i\}$

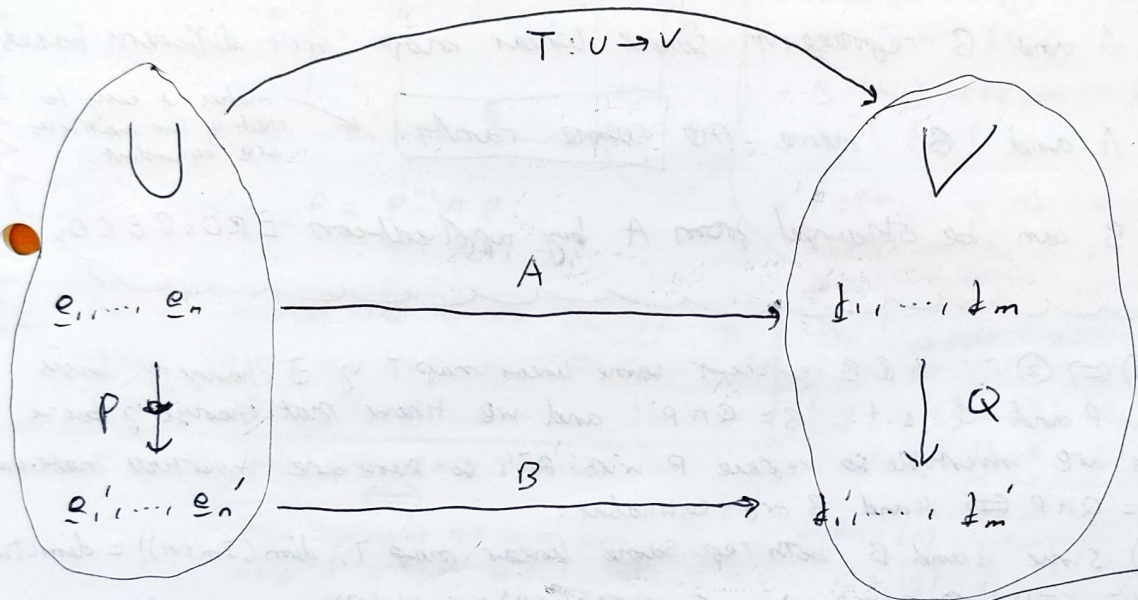
Let the $m \times m$ matrix $Q = \{q_{ij}\}$ be the change of basis matrix from $\{d_i\}$ to $\{d'_i\}$

Aim: Find relationship between A and B wrt. the change of basis matrices P and Q

$$\begin{array}{ccc} U & & V \\ e_1, \dots, e_n & \xrightarrow{A} & d_1, \dots, d_m \\ \downarrow P & & \downarrow Q \\ e'_1, \dots, e'_n & \xrightarrow{B} & d'_1, \dots, d'_m \end{array}$$

$$B = QAP^{-1}$$

rough sketch of change
relationships between
change of bases matrices
and matrices for linear
transformations w.r.t different
bases



Theorem: $BP = QA \Leftrightarrow B = QAP^{-1}$

Proof: ① $I_U: U \xrightarrow{P} U$
 $e_1, \dots, e_n \quad e'_1, \dots, e'_n$

$T: U \xrightarrow{B} V$
 $e_1, \dots, e_n \quad f'_1, \dots, f'_m$

$\therefore T \circ I_U: U \xrightarrow{BP} V$
 $e_1, \dots, e_n \quad f'_1, \dots, f'_m$

② $T: U \xrightarrow{A} V$
 $e_1, \dots, e_n \quad f_1, \dots, f_m$

$I_V: V \xrightarrow{Q} V$
 $f_1, \dots, f_m \quad f'_1, \dots, f'_m$

$\therefore I_V \circ T: U \xrightarrow{QA} V$
 $e_1, \dots, e_n \quad f'_1, \dots, f'_m$

so combining ① and ②

$\Rightarrow BP = QA$

two $m \times n$ matrices A and B represent the same linear map from an n -dimensional vector space to an m -dimensional vector space

$\Leftrightarrow$
 there exist invertible $n \times n$ and $m \times m$ matrices P and Q with
 $B = QAP$

Equivalent matrices

Def: Two $m \times n$ matrices A and B are equivalent if there exist invertible P and Q with $B = QAP$

equivalent conditions:

- ① A and B are equivalent
- ② A and B represent same linear map wrt different bases
- ③ A and B have the same rank ← makes it easy to check if two matrices are equivalent.
- ④ B can be obtained from A by application EROs & ECOs?

Proof: ① $\Leftrightarrow$ ② $\because$ A & B represent same linear map T if $\exists$ change of basis matrices P and Q s.t. $B = QAP^{-1}$ and we know that change of basis matrices are invertible so replace P with P^{-1} so there are invertible matrices with $B = QAP \Leftrightarrow A$ and B are equivalent.

② $\Rightarrow$ ③ Since A and B both rep. same linear map T , $\dim(\text{Im}(A)) = \dim(\text{Im}(B)) = \dim(\text{Im}(T))$ otherwise $\times$ so $\text{rank}(A) = \text{rank}(B)$.

③ $\Rightarrow$ ④ $\because$ as A and B both have rank s , can ^{both} be brought into block form

$$E_s = \left(\begin{array}{c|c} I_s & O_{s, n-s} \\ \hline O_{m-s, s} & O_{m-s, n-s} \end{array} \right)$$

by EROs and ECOs. As EROs & ECOs invertible, $A \xrightarrow{\text{operations}} E_s \xrightarrow{\text{operations}} B$

④ $\Rightarrow$ ① can write these transformations operations as a product of elementary row & column operations so $\exists$ elementary row matrices $R_1, \dots, R_r$ and EC matrices $C_1, \dots, C_s$ s.t. $B = R_r \dots R_1 A C_1 \dots C_s$ so take the product Q it is invertible so $\exists$ invertible P and Q s.t. $B = QAP$. $\square$

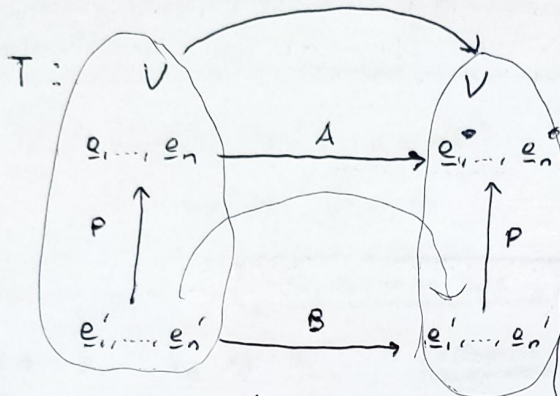
note: any $m \times n$ matrix A is equivalent to the matrix E_s above where $s = \text{rank}(A)$
The form E_s is a canonical form for $m \times n$ matrices under equivalence.

Similar $\Rightarrow$ equivalent

equivalent $\nRightarrow$ Similar

Similar Matrices

V vector space over field $\mathbb{K}$. $\dim(V) = n$. $T: V \rightarrow V$ is the linear map



Setup:

- $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ are two bases of V .
- $A = (a_{ij})$ is the matrix for T wrt $\{e_i\}$
- $B = (b_{ij})$ is the matrix for T wrt $\{e'_i\}$
- $P = (p_{ij})$ is the change of base matrix from $\{e'_i\}$ to $\{e_i\}$

Then: clearly, $B = P^{-1}AP$

Def: two $n \times n$ matrices over $\mathbb{K}$ are similar if there exists an $n \times n$ invertible matrix P with $B = P^{-1}AP$

In other words: two matrices are similar $\iff$ they represent the same linear map $T: V \rightarrow V$ w.r.t. ~~two~~ different bases of V

Similar matrices have the same characteristic polynomial & hence the same eigenvalues

Proof: Let A and B be similar matrices. Then $\exists P$ [invertible] with $B = P^{-1}AP$ so

$$\begin{aligned} \det(B - xI_n) &= \det(P^{-1}AP - xI_n) \\ &= \det(P^{-1}(A - xI_n)P) \\ &= \det(P^{-1}) \det(A - xI_n) \det(P) \\ &= \frac{\det P}{\det(P)} \det(A - xI_n) \\ &= \det(A - xI_n) \end{aligned}$$

so A and B have the same characteristic polynomial $\implies$ same eigenvalues

E.g. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where

$$T(x) = A_2 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \neq$$

in basis (of eigenvectors)

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for $\mathbb{R}^3$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and computing change of basis matrix from new to old

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$P^{-1}A_2P = B \text{ of required so } A_2 \text{ \& } B \text{ similar.}$$

Note: the different matrices corresponding to a linear map T are all similar so they all have the same characteristic equation so can refer to it as the characteristic polynomial of T .

$\Downarrow$
hard to find a E_s like row column reduced canonical form like we do for commutative matrices.

Similar vs Equivalent matrices

$$T: U \rightarrow V$$

Two $m \times n$ matrices are equivalent if $\exists$ invertible P and Q with $B = QAP$

$$T: V \rightarrow V$$

Two $n \times n$ matrices over $\mathbb{K}$ are similar if $\exists$ $n \times n$ invertible matrix P with $B = P^{-1}AP$

Def: $\underline{v} = (a_1, \dots, a_n)$, $\underline{w} = (b_1, \dots, b_n)$, $\underline{v}, \underline{w} \in \mathbb{R}^n$.

$$\underline{v} \cdot \underline{w} = \sum_{i=1}^n a_i b_i \quad [\text{scalar product}]$$

A & B have same characteristic polynomial & same eigenvalues.

orthonormal basis

A basis $\underline{b}_1, \dots, \underline{b}_n$ of $\mathbb{R}^n$ is orthonormal if

$$\textcircled{1} \quad \underline{b}_i \cdot \underline{b}_i = 1 \quad \text{for } 1 \leq i \leq n$$

$$\textcircled{2} \quad \underline{b}_i \cdot \underline{b}_j = 0 \quad \text{for } 1 \leq i, j \leq n \text{ and } i \neq j$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

↑
standard basis for $\mathbb{R}^3$ is orthonormal

Symmetric matrices

$n \times n$ matrix A symmetric if

$$A^T = A$$

orthogonal matrices

$n \times n$ matrix A is orthogonal if

$$A^T = A^{-1}$$

$$\Leftrightarrow AA^T = A^T A = I_n$$

and of:

A orthonormal matrix

$$A \in M_{2 \times 2}(\mathbb{R})$$

$$\Leftrightarrow$$

rows $\underline{e}_1, \dots, \underline{e}_n$ of A

form an

orthonormal basis of $\mathbb{R}^n$

$$\Leftrightarrow$$

columns $\underline{e}_1, \dots, \underline{e}_n$ of A

form an orthonormal basis of $\mathbb{R}^n$

A is a real symmetric matrix



A has only real eigenvalues

Pf: $\det(A - \lambda I_n)$ has a root $\lambda \in \mathbb{C}$ which is an eigenvalue of A

$$A\underline{v} = \lambda \underline{v} \quad \text{and transpose it} \quad \underline{v}^T A = \lambda \underline{v}^T \quad (3)$$

$$A\underline{v} = \bar{\lambda} \underline{v} \quad (4) \quad \text{and transpose it}$$

multiply (3) by $\underline{v}^T$ on the right.
multiply (4) by $\underline{v}^T$ on the left.] compare: set $\lambda = \bar{\lambda}$

A is a real symmetric matrix. λ_1, λ_2 are two distinct eigenvalues of A [corresponding $\underline{v}_1, \underline{v}_2$]



$$\underline{v}_1 \cdot \underline{v}_2 = 0$$

let A be a real symmetric matrix ($n \times n$).
Then there exists a real orthogonal matrix P with $P^{-1}AP (= P^T A P)$ diagonal.

change of basis - matrix of normalized eigenvectors.

Pf: set an orthonormal basis of $\mathbb{R}^n$ [eigenvectors]
so $P^{-1}AP$ diagonal Entries $(P^{-1}AP)_{ii} = \lambda_i$