Analysis II - MA133

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Difficulty of proof:
- Easy: *
- Hard?: **
- what??: ***

□ = Def:
☼ = Thm
○ = corollary

Chapter 1

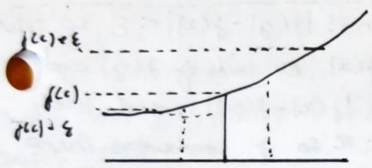
[continous functions]

Def: continuity

A function $f: I \to \mathbb{R}$ is $\boxed{\text{continous}}$ at c [f defined on an interval I containing c] if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $x \in I$, 0

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Function is continous on I if it is continous at each point of I



Thm: Sequential continuity

Let $f: I \to \mathbb{R}$ be ~~continuous~~ defined on the interval I and suppose that $c \in I$. Then

f is $\boxed{\text{continous}}$ at c $\iff$ for every sequence (x_n) of points which converge to c, $f(x_n) \to f(c)$ as $n \to \infty$

Pf: (*) [iff, contrapositive, build a sequence]

Thm: Algebra of continous functions

Let $f, g: I \to \mathbb{R}$ be defined on the interval I and continous at $c \in I$. Then

1. $f + g$ continous at c
2. $f \cdot g$ continous at c
3. If $g(c) \neq 0$, f/g continous at c

Pf: (*) use sequential continuity & analysis I

Cor: Continuity of polynomials & rational functions

- If p is a polynomial, then p is continous at every point of $\mathbb{R}$
- If p/q is a ratio of two poly. then it is continous $\forall x \in \mathbb{R}, q \neq 0$

Pf: apply algebra of continous f. *

Thm: composition of continous functions

Let $f: I \to \mathbb{R}$ be defined on the interval I
Let $g: J \to I$ be defined on the interval J
If g is continous at c, f continous at $g(c)$ then $f \circ g$ is continous at c

Pf: (*) use ~~algebra~~ sequential continuity and apply f to it

Thm: Intermediate Value Theorem

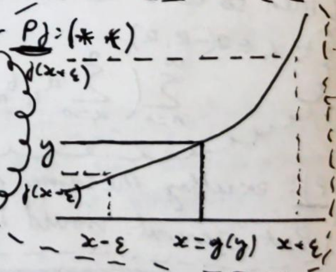
Let $f: [a, b] \to \mathbb{R}$ be continous and suppose $u \in (f(a), f(b))$, then $\exists c \in (a, b)$ s.t. $f(c) = u$

Pf: (***) define A carefully and use both properties of least upper bound to find contradictions

Thm: Existance of Inverses

Let $f: [a, b] \to \mathbb{R}$ be continous and strictly increasing. Then f has an inverse defined on its range and f^{-1} is continous

Pf: (**)



Cor: continous image of an interval

If $f: I \to \mathbb{R}$ is continous on the interval I, then its range is an interval

Pf: IVT

Thm: Boundedness of continous functions

Let $f: [a, b] \to \mathbb{R}$ be continous. Then f is bounded

Pf: contradiction, bolzano-weirstrauss

Thm: Attainment of Bounds

Let $f: [a, b] \to \mathbb{R}$ be continous, then f has a maximum & a minimum attained on the interval

Pf: take a least upper bound on the set of image points for f. Suppose there is no point ~~in~~ in $[a, b]$ that achieves the maximum.
Define a new strictly positive increasing function, g in terms of maximum. Find the inversed obtain a contradiction

Chapter 2: Power Series I

Def: Power Series

A series of the form
$$\boxed{\sum_{n=0}^{\infty} a_n x^n} \text{ or } \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
[centred at x_0]

Thm: Radius of Convergence I

Let $\sum_0^\infty a_n x^n$ be a power series with $\sum a_n t^n$ convergent. Then
$$\sum_0^\infty a_n x^n$$
converges absolutely $\forall x$ with $|x| < |t|$

Pf: Boundedness, inequalities, geometric series

use the ratio test to find R

Thm: Radius of Convergence II

Let $\sum_0^\infty a_n x^n$ be a power series. Then one of following holds

- series converges only if $x = 0$
- series converges $\forall x \in \mathbb{R}$
- $\exists R \in \mathbb{R}, R > 0$ s.t. series conv. for $|x| < R$, diverges for $|x| > R$.

$\boxed{\text{Radius of convergence}}$ is R

in ①, $R = 0$, in ②, $R = \infty$

Pf: $R = \sup \{ |t| : \sum a_n t^n \text{ converges} \}$

Cor: Let $\sum a_n x^n$ be a pwr series with RoC R. Then $\sum |a_n| x^n$ also has RoC R.

Absolute Series

Would move onto the derivative now, but we apply current theory to the exponential to maintain audience interest!

Thm: Continuity of Power Series

Let $\sum_0^\infty a_n x^n$ be a power series w/ RoC R. Then the function
$$x \longmapsto \sum_0^\infty a_n x^n$$
is continous on the interval $(-R, R)$

Pf: (***) WTS $|f(y) - f(x)| < \varepsilon$ so split tails of $f(x)$, tails of $f(y)$ and body of $|f(x) - f(y)|$, and pick a $|t| < R$ so f converges there

~~Def: If the series~~

Pf: Use the ratio test!
$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \to \frac{x}{n+1} \to 0$$
so series converges

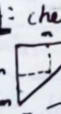
Def: The Exponential

If $x \in \mathbb{R}$, the series
$$\boxed{1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots}$$
converges, we call the sum $\boxed{\exp x}$

Thm: The characteristic property of $\exp x$

If $x, y \in \mathbb{R}$, then $\exp(x+y) = \exp(x)\exp(y)$

Pf: check $e^{x+y} - e^x e^y \to 0$. Use binomial theorem and we show that the corner pieces are [illegible] out



Thm: The Product of Power Series

If $\sum a_n x^n$ and $\sum b_n x^n$ converge for $x \in (-R, R)$ then so does the series $\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$ and $\forall x \in (-R, R)$
$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n = \left(\sum_{i=0}^{\infty} a_i x^i \right) \left(\sum_{j=0}^{\infty} b_j x^j \right)$$

Pf: exactly the same method as $\exp(x+y) = \exp(x)\exp(y)$

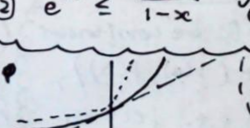
But general. Would be great to know!

Thm: Inequalities for the exponential

The following hold:

① $1 + x \le e^x \quad \forall x \in \mathbb{R}$

② $e^x \le \frac{1}{1-x}$ if $x < 1$



Pf: For $x > 0$, use powerseries & obvious inequalities. For $x < 0$, use e^{-x}

Pf: sps $x < y$ + inequality ①

Cor: The exponential increases

$\exp x$ is continous & strictly increasing so...

Thm: The logarithm

There is a continous strictly increasing function $x \longmapsto \log x$ defined on $(0, \infty)$ satisfying

- $e^{\log x} = x \quad \forall x > 0, x \in \mathbb{R}$
- $\log(e^y) = y \quad \forall y \in \mathbb{R}$
- $\log(uv) = \log(u) + \log(v) \quad \forall u, v \in \mathbb{R}$

Pf: $e^{\log u + \log v} = e^{\log u} e^{\log v} = uv$ & apply logs.

Thm: tangent to log

If $x > 0$, $\log x \le x - 1$

Def: Powers if $x > 0$, $p \in \mathbb{R}$ define
$$\boxed{x^p = \exp(p \log x)}$$
we have the usual rules:

① $n \in \mathbb{N}_0$, $x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$

② $x^{p+q} = x^p x^q \quad \forall x > 0, \; p, q \in \mathbb{R}$

③ $\log(x^p) = p \log(x) \quad \forall x > 0, \; p \in \mathbb{R}$

④ $x^{pq} = (x^p)^q \quad \forall x > 0, \; p, q \in \mathbb{R}$

⑤ $\exp(p) = e^p \quad \forall p \in \mathbb{R}$

Chapter 3: Limits & the Derivative

Def: Limits of functions

Let I be an open interval, $c \in I$ and f a real valued function defined on I ~~exep~~ except possibly at c. We say

$$\boxed{\lim_{x \to c} f(x) = L}$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

so we don't care about $f(c)$!

Lemma: limits and continuity

If $f : I \to \mathbb{R}$ is defined on the open interval I and $c \in I$, then f continuous at c

$$\Updownarrow$$

$$\lim_{x \to c} f(x) = f(c)$$

Pf: shuffle around the definitions of limits & continuity

Thm: Continuous & sequential limits

If $f : I \setminus \{c\} \to \mathbb{R}$ is defined on the interval I except at $c \in I$, then

$\lim_{x \to c} f(x) = L \iff$ for every sequence (x_n) in $I \setminus \{c\}$ with $x_n \to c$, we have $f(x_n) \to L$

Pf: in HW! (**)

Thm: Algebra of limits

If $f, g : I \setminus \{c\} \to \mathbb{R}$ are defined on the interval I except at $c \in I$ and $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then

1. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} (f(x)) + \lim_{x \to c} (g(x))$
2. $\lim_{x \to c} (f(x) g(x)) = \lim_{x \to c} (f(x)) \lim_{x \to c} (g(x))$
3. If $\lim_{x \to c} (g(x)) \neq 0$ then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

Pf: convert to a sequences question using *!

Def: One sided limits

Let f be a real valued function defined on the open interval (c, d). Then

$$\boxed{\lim_{x \to c^+} f(x) = L}$$

if for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $c < x < c + \delta$, then $|f(x) - L| < \varepsilon$

"The limit of $f(x)$ as x approaches from the right is L"

Def: Infinite limits

If $f : I \setminus \{c\} \to \mathbb{R}$ is defined on an interval I except perhaps at c, we write

$$\boxed{\lim_{x \to c} f(x) = \infty}$$

if $\forall M > 0$, $\exists \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $f(x) > M$.

Def: Limits at infinity

If $f : \mathbb{R} \to \mathbb{R}$, we write

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

if for every $\varepsilon > 0$ $\exists N$ s.t. if $x > N \implies |f(x) - L| < \varepsilon$

Def: The Derivative

Suppose $f : I \to \mathbb{R}$ is defined on the open interval I and $c \in I$. We say f is **differentiable** at c if

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists. We call this limit $f'(c)$

Note: can rewrite derivative as the following

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Lemma: Differentiability and continuity

If I is an open interval $f : I \to \mathbb{R}$ is differentiable at $c \in I$. Then f is continuous at c

Pf: use limit version of continuity. $f(x) - f(c) \to 0$ using what you know

Lemma: The derivative of the monomials

$n \in \mathbb{N}$, then $\frac{d}{dx}(x^n) = n x^{n-1}$

Pf: induction!

Thm: The Sum & Product Rules

Suppose $f, g : I \to \mathbb{R}$ are defined on the open interval I and are differentiable at $c \in I$. Then $f + g$ and fg differentiable at c and

$(f+g)'(c) = f'(c) + g'(c)$

$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

Lemma: Local linearisation

Suppose I is an open interval, $f: I \to \mathbb{R}$ and $c \in I$. Then

f is differentiable at c $\iff$ $\exists A \in \mathbb{R}$, function $\mathcal{E}$ with properties that
① $\forall x \in I$ $f(x) - f(c) = A(x-c) + \mathcal{E}(x)(x-c)$
② $\mathcal{E}(c) = 0$
③ $\mathcal{E}$ continuous at c
$\mathcal{E}(x) \to 0$ as $x \to c$
$A = f'(c)$ if this happens

Pf: Define $\mathcal{E}$ carefully & use everything!

Thm: Chain Rule

Suppose I and J are open intervals $f: I \to \mathbb{R}$, $g: J \to I$, that g is differentiable at c and f is differentiable at $g(c)$. Then the composition $f \circ g$ is differentiable at c and

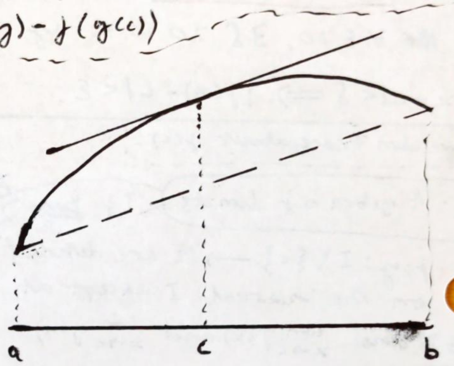
$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c)$$

Pf: (**) use the local linearisation lemma and consider $f(y) - f(g(c))$

Thm: Mean Value Theorem

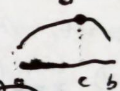
Suppose $f: [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then $\exists c \in (a,b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Pf: adapt Rolle's theorem with
$g(x) = f(x) - x\dfrac{f(b) - f(a)}{b-a}$
$g(b) - g(a) = 0$ etc.

Thm: Rolle's theorem



Suppose $f: [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and that $f(a) = f(b)$. Then $\exists c \in (a,b)$ where $f'(c) = 0$

uses Bolzano Weierstrass

Pf: Assume larger than $f(a)$ somewhere attains maximum at c. get conditions on $\frac{f(x)-f(c)}{x-c}$ for $x > c$, and $x < c$ to deduce $f'(c) = 0$

Cor: functions with positive derivative

If $f: I \to \mathbb{R}$ is differentiable on the open interval I and $f'(x) > 0$ $\forall x \in I$, then f is strictly increasing on I

Pf: Assume $a < b$ but $f(a) \geq f(b)$ contradicting MVT as $f'(c) > 0$ $\forall c \in I$

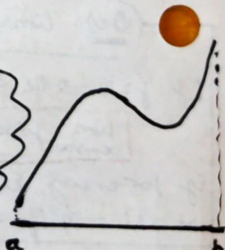
Cor: functions with zero derivative

$f'(x) = 0$ $\forall x \in I \implies f$ constant on I

Pf: direct from MVT!

Thm: Extrema & derivatives

Suppose $f: [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b). Then f obtains max/min either at points in (a,b) where $f' = 0$ or at ends a or b



Note: If max occurs at $c \implies$ ① $f'(c) = 0$ ② $f''(c) < 0$] useful info

Uniqueness of solutions to a DE

Method: If solutions are $f(x) = Ae^x$ then set $g(x) = e^{-x} f(x)$. [rearrange for constant]. $g'(x) = 0 \implies g$ is a constant by MVT and rearrange for f

Derivatives of Inverses

$f(g(x)) = x \implies f'(g(x))g'(x) = 1 \implies \boxed{g'(x) = \dfrac{1}{f'(g(x))}}$

Thm: Derivatives of inverses

Let $f: (a,b) \to \mathbb{R}$ be differentiable with positive derivative. Then $g = f^{-1}$ is differentiable and

$$g'(x) = \frac{1}{f'(g(x))}$$

Pf: (**) Lots of variables & lots of switching. Know where heading!

Chapter 4: Power Series II

Lemma: The Differentiability of power series I

Let $\sum a_n x^n$ be a power series with radius of convergence R. Then the series $\sum n a_n x^{n-1}$ has the same radius of convergence

Pf: consider $\frac{f(y)-f(x)}{y-x}$ and use inequalities

Thm: The Differentiability of Power series II

Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence R. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Pf: Similar to pf of continuity $\frac{f(y)-f(x)}{y-x} - f'(x) < \varepsilon$ and use body and tail principle (***)

Cor: Derivative of exponential

$$\exp'(x) = \exp(x)$$

Pf: term by term on series

Cor: char property of exp

$\forall x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x)\exp(y)$

Pf: consider $g(x) = \exp(x)\exp(z-x)$ differentiate, zero. $g(0) = \exp(z)$ and sub in with $z = x+y$

Diff eqns can set $y = \sum a_n x^n$ and sub into DEs to show solutions

If I know f, how find coefficients?

Soln: sub in zero & differentiate each time

Def: The Trig functions

ratio test — convergence; cosine — even; sine — odd; diff. term by term — derivatives

$\forall x \in \mathbb{R}$ we define

$$\boxed{\cos x} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots + (-1)^k \frac{x^{2k}}{(2k)!} + \dots$$

$$\boxed{\sin x} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots$$

Thm: Addition formula for trig functions

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

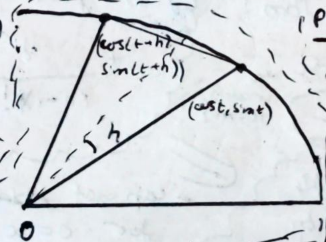
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Pf: Let $f(z) = \cos x \cos(z-x) - \sin x \sin(z-x)$. $f'(z) = 0$, $\Rightarrow$ MVT $\Rightarrow$ constant. Sub in zero find

Cor: The circular property

$\forall x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$

Pf: Set $y = -x$ in addition formula for $\cos(x-y)$



WTS: as $t \uparrow$, $(\cos t, \sin t)$ traces out a circle at rate 1. Let $L(t)$ be length of circular arc from $(1,0)$ to $(\cos t, \sin t)$. WTS $L(t) = t$. By MVT $\Leftrightarrow$ $L'(t) = 1 \ \forall t$

Pf: Straight line distance means WTS $\forall t$

$$\lim_{h \to 0^+} \frac{1}{h} \sqrt{(\cos(t+h) - \cos(t))^2 + (\sin(t+h) - \sin t)^2} = 1$$

Note by geometry, straight line distance $= 2\sin\left(\frac{h}{2}\right)$

[prove by repeated use of addition formula] so

$$\lim_{h \to 0^+} \frac{2\sin\left(\frac{h}{2}\right)}{h} = \lim_{p \to 0^+} \frac{\sin(p)}{p} = 1 \quad [\text{derivative of } \sin \text{ at } 0]$$

The complex exponential "our jewel"

$$e^{it} = \cos t + i \sin t$$

Same structure as exponential property & trig addition formulae

No simple ~~form~~ power series for $\tan x$. It doesn't exist!

This is wi maths for you!!!

Chapter 5: Taylor's Theorem

Thm: Cauchy's Mean value theorem

If $f, g : [a,b] \to \mathbb{R}$ are continuous, differentiable on (a,b) and $g'(t) \neq 0 \ \forall t \in (a,b)$. Then $\exists t \in (a,b)$ s.t.

$$\frac{f'(t)}{g'(t)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Pf: consider $x \mapsto h(x) = f(x)(g(b) - g(a)) - (f(b) - f(a))g(x)$

Note: $h(a) = h(b)$ so by Rolle's Theorem, place $t \in (a,b)$ where $h'(t) = 0$.

Sub in and rearrange. $g'(t) \neq 0$

extend pointwise
form domains

Another version at infinity!

[extended MVT really]

Thm: L'Hopital's Rule

If $f, g: I \to \mathbb{R}$ are differentiable on the open interval I containing c and $f(c) = g(c) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Pf: Apply Cauchy's MVT and take limit as $x \to c$. [t forced to c too]

Thm: L'Hopital's Rule at infinity

If $f, g: \mathbb{R} \to \mathbb{R}$ are differentiable and

$\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$

or

$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$ [provided 2nd limit exists]

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$ is too imprecise for uni! need to express the error in terms of f

Thm: Taylor's Thm, Lagrange Remainder

If $f: I \to \mathbb{R}$ is n times differentiable on $I \ni a, b$ then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \dots$$
$$\dots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(t)}{n!}(b-a)^n$$

for some $t \in (a,b)$ — error term — function of t — $k=0$

Pf: (***) Define $h(x) = ?$ and apply Rolle's thm and induction, then sub in b etc.

Applications:

- error in functions
- Inequalities proved

Pf: Not on exam, mixture of both thms

Thm: Taylor's Theorem, trickier pf

If $f: I \to \mathbb{R}$ is n times differentiable on the open interval I containing a, b and $0 \le k \le n-1$ then

$$f(b) = f(a) + f'(a)(b-a) + \dots + \frac{f^{(n-1)}}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(t)}{(n-1)!\,(n-k)}(b-t)^k(b-a)^{n-k}$$

Pf: use trickier $k = n-1$

Thm: Taylor's Thm, Cauchy Remainder

If $f: I \to \mathbb{R}$ is n times differentiable on the open interval I containing a and b, then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \dots$$
$$\dots + \frac{f^{(n)}(t)}{(n-1)!}(b-a)^{n-1}(b-a)$$

- Can get $\log x$ to converge for $0 < x < 1$ using Cauchy remainder.

Appendix: Radius of Convergence Formula

Thm: Power series $\sum a_n x^n$. Consider the sequence of terms $|a_n|^{1/n}$. $\limsup(|a_n|^{1/n}) = L$. [$L>0$]

Power series has a radius of convergence

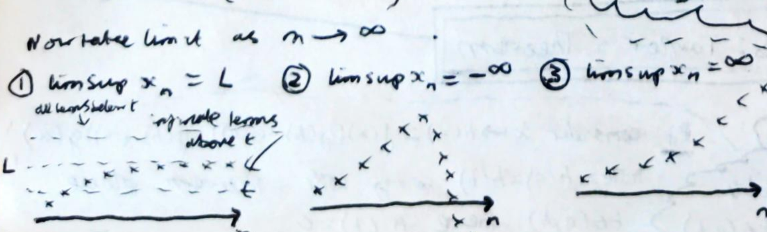
$$\frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

Supremum of the tails of a sequence (x_n)

$$u_m = \sup\{x_n : n \geq m\}$$

Now take limit as $n \to \infty$

① $\limsup x_n = L$ ② $\limsup x_n = -\infty$ ③ $\limsup x_n = \infty$



[bounded above] [bounded above] [unbounded]

[limsup = how high sequence reaches over & over again]

- If $|x| > \frac{1}{L}$, $|a_n x^n| \geq 1 \Rightarrow$ diverges
- If $|x| < \frac{1}{L}$, $|a_n x^n| \le (t|x|)^n < \infty$ convergent

so radius of convergence is $1/L$.