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Probability A - ST111

Chapter 1

Sample space Ω
of an experiment is the set of all possible outcomes

event A
event A is $A \subseteq \Omega$

Note: create mutually exclusive events like B and $A \setminus B$.

probability measure
A function $\mathbb{P}: 2^{\Omega} \to [0,1]$ s.t.
① $\mathbb{P}(\emptyset)=0$, $\mathbb{P}(\Omega)=1$
② For every pairwise disjoint $A_1, A_2, \dots \subseteq \Omega$

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$$

uniform probability measure
every pair of singletons are equally likely events

$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ [can use area/volume too]

Inclusion-Exclusion
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
Pf: consider A, $B \setminus A$ and $B \setminus A$, $A \cap B$ separately & apply additivity

Formula

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{n+1} p_k$$

where for $1 \le k \le n$

$$p_k = \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k})$$

Chapter 2

Sequence of k elements in Ω is a function from $\{1, \dots, k\}$ to Ω
$j: \{1, \dots, k\} \to \Omega$
i.e. $(j(1), j(2), \dots, j(k))$

If j injective, we have distinct elements.

n^k sequences of k elements in $|\Omega| = n$. Use the principle of multiplication $j(i) \in \Omega$ and n choices for each of the k elements

$\frac{n!}{(n-k)!}$ sequences of k distinct elements in $|\Omega| = n$
Pf: induction

number of subsets having exactly k elements is ${}^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$n_1 + n_2 = n$
population n: n_1 | n_2

A = event sample of size k contains k_i individuals of type i

① without replacement:

$$\mathbb{P}(A) = \frac{\binom{n_1}{k_1} \times \binom{n_2}{k_2}}{\binom{n}{k}}$$ hypergeometric

② with replacement:

$$\binom{k}{k_1}\left(\frac{n_1}{n}\right)^{k_1}\left(\frac{n_2}{n}\right)^{k_2}$$ binomial

Chapter 3

Conditional probabilities

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Partition
collection of events $A_1, \dots, A_n$ partitions Ω a.s.s.
① $\bigcup_{i=1}^{n} A_i = \Omega$ ② $A_i \cap A_j = \emptyset \ \forall i \neq j$
③ $A_i \neq \emptyset \ \forall i$

multiplication rule
$A_1, \dots, A_n$ finite sequence of events

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1) \times \mathbb{P}(A_3 \mid A_1 \cap A_2) \times \dots \times \mathbb{P}(A_n \mid A_1 \cap \dots \cap A_{n-1})$$

Law of Total Probability
$A_1, \dots, A_n$ is a partition of Ω. $\mathbb{P}(A_i) > 0 \ \forall i$

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B \mid A_i)$$

Pf: $\bigcup_{i=1}^{n} (A_i \cap B)$ union of disjoint events, apply $\mathbb{P}$ measure

Bayes' Theorem
Let $A_1, \dots, A_n$ be a partition of Ω. B further event

$$\mathbb{P}(A_j \mid B) = \frac{\mathbb{P}(B \mid A_j)\mathbb{P}(A_j)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A_j)\mathbb{P}(A_j)}{\sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B \mid A_i)}$$

Independence
- A, B independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \iff \mathbb{P}(A \mid B) = \mathbb{P}(A)$
- $A_1, A_2, \dots$ independent if for every choice of distinct integers $i_1, i_2, \dots, i_n \in \{1, \dots\}$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \dots \mathbb{P}(A_{i_n})$$

Chapter 4

$A_1, \dots, A_n$ independent events $\mathbb{P}(A_i) = p \ \forall i$
$\mathbb{P}(k$ exactly occur$)$

$$p(k, n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

binomial
Pf: $E_k = \bigcup_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = k}} \bigcap_{i \in S} A_i \cap \bigcap_{i \notin S} A_i^c$ and use $\mathbb{P}$ axioms & independence

Can let X be the event k occur...

Weak Law of Large Numbers
Let $X_n \sim \text{Bin}(n, p)$, then for any $\varepsilon > 0$ [tail gets small as $n \to \infty$]

$$\mathbb{P}\left(\left|\frac{X_n}{n} - p\right| > \varepsilon\right) \to 0$$

Poisson distribution
$X \sim \text{Poi}(\lambda)$ if $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

Stirling's formula

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1$$

Gaussian Distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

standard density function

CLT

$$\mathbb{P}(z_1 < X < z_2) = \int_{z_1}^{z_2} \phi(z)\,dz$$

$X \sim \text{Normal}(\mu, \sigma^2)$

Chapter 5

$X_n \sim \text{Bin}(n, p = \frac{\lambda}{n})$ $\mathbb{P}(X_n = k) \to \frac{\lambda^k}{k!} e^{-\lambda}$

Pf: using $\frac{\mathbb{P}(X_n = k)}{\mathbb{P}(X_n = k-1)} \to \frac{\lambda}{k}$ and plug into

$$\mathbb{P}(X_n = k) = \mathbb{P}(X_n = 0) \times \frac{\mathbb{P}(X_n = 1)}{\mathbb{P}(X_n = 0)} \times \dots \times \frac{\mathbb{P}(X_n = k)}{\mathbb{P}(X_n = k-1)}$$

Poisson