Introduction to Partial Differential Equations
Summary

Note: 2 days till exam & optional choice question so will leave out details of chapter 6

Chapter 1 - Introduction

① PDEs & differential operators: can write a PDE as $F(x_1, \dots, x_n, u, \partial_{x_1} u, \dots) = 0$ for some $F : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^m$. $x = (x_1, \dots, x_n)$, k is # partial derivatives of any order, m is # eqns (system if $m > 1$). $\mathcal{L}$ is the associated differential operator. PDE then reads $\mathcal{L}(u) = r(x)$

- PDE is linear if $\mathcal{L}$ is linear. $\forall u, v$ functions, $c \in \mathbb{R}$

$$\mathcal{L}(u + cv) = \mathcal{L}(u) + c\,\mathcal{L}(v)$$

- PDE is homogeneous if $r = 0$ so PDE reads $\mathcal{L}(u) = 0$

- Order of the PDE is the order of the highest (possibly mixed) partial derivative appearing in the PDE. E.g. $\partial_{x_1 x_2}$ is order 2

② Well posedness of PDE problems: A PDE is well posed if it has a unique solution that depends continuously on the data:

$$\text{well posedness} = \underset{\text{stability}}{\text{existence}} + \text{uniqueness} + \text{stability}$$

data = values & parameters entering problem

Chapter 2 - First Order Equations

③ Transport Equation, initial condition, Cauchy problem: Find $u(x,t)$ in $t \in [0,T]$ $T \in (0, \infty)$, $v(x,t)$ is 'velocity' of system. Assume $\Phi : \mathbb{R} \to \mathbb{R}$ w/ $u(x_0, 0) = \Phi(x_0)$ $\forall x_0 \in \mathbb{R}$.

$$\boxed{\partial_t u + v\,\partial_x u = 0}$$

follows a particle in time

④ Characteristics: Define $\xi(t) (= x)$ s.t. $\xi'(t) = v(\xi(t), t)$ with $\xi(0) = x_0$
Then

$$\frac{d}{dt} u(\xi(t), t) = u_x \cdot \xi'(t) + u_t = u_t + v u_x = 0$$

u doesn't change along characteristic

So if $\Phi(x_0) = u(x_0, 0)$, $u(x,t) = u(\xi(t), t) = u(\xi(0), 0) = \Phi(x_0)$

Example: $v(x,t) = x$, $\Phi(x_0) = x_0^2 - 2x_0$. Then IVP: $\left\{ \begin{array}{l} \xi'(t) = v(\xi(t), t) = \xi(t) \\ \xi(0) = x_0 \end{array} \right\}$
So $\xi(t) = x_0 e^t$. As characteristic constant,

$$u(x,t) = u(\xi(t), t) = u(\xi(0), 0) = \Phi(x_0) = x^2 e^{-2t} - 2x e^{-t}$$

⑤ Source terms (inhomogeneous transport equation): a $S(x, t, u(x,t))$ now present:

$$\boxed{\partial_t u + v\,\partial_x u = S(u)}$$

Applying characteristics again, $\frac{d}{dt} u(\xi(t), t) = \partial_t u \cdot \xi'(t) + \partial_x u = v\,\partial_t u + \partial_x u = \dots$
$\dots = S(\xi(t), t, u(\xi(t), t))$

So need to solve an ODE to learn solution. Initial condition is $u(\xi(0), 0) = \Phi(x_0)$

Example: $v(x,t) = x$, $S(u) = -u$, $u(x_0, 0) = \cos(x_0)$. $\xi'(t) = v(\xi(t), t)$, $\xi(0) = x_0$
So characteristic $\xi(t) = x_0 e^t$. Then

(*)

$\frac{d}{dt} u(\xi(t), t) = \partial_x u \cdot \xi'(t) + \partial_t u = -u(\xi(t), t)$

so solve $\frac{d}{dt} u(\xi(t),t) = -u(\xi(t),t)$, $u(x_0, 0) = \cos(x_0)$ $x_0 = \xi(0)$

Solution is $u(\xi(t),t) = \cos(x_0) e^{-t}$. For any point (x,t), we have $x = \xi(t)$ if $x_0 = x e^{-t}$ (*) so

$$\boxed{u(x,t) = \cos(x e^{-t}) e^{-t}}$$

solution

⑥ Non-linear equations: Try $\partial_t u + v(u) \partial_x u = 0$ with $v(u) = u$. can still apply method of characteristics. If characteristics cross, this fails...

$$\partial_t u + u \partial_x u = 0 \quad \text{with } u(x,t) \text{ solution}$$

Then $\xi'(t) = u(\xi(t), t)$, $\xi(0) = x_0$ [characteristic depends on u]

$$\frac{d}{dt} u = \partial_x u \cdot \xi'(t) + \partial_t u = 0 \quad \text{so can solve}$$

$$\xi'(t) = u(\xi(0), 0) = \Phi(x_0) \overset{\text{solving}}{\Rightarrow} \xi(t) = \Phi(x_0) t + x_0$$

Chapter 3 – The wave Equation

This inspires us to try characteristics...

⑦ Characteristic coordinates, general solution: we want to solve wave eqn in one spatial dimension. $\boxed{\partial_{tt} u = c^2 \partial_{xx} u}$ $\square = (\underbrace{\partial_t + c\partial_x}_{\text{transport eqn}})(\underbrace{\partial_t - c\partial_x}_{\text{transport eqn}})$

consider the characteristic coordinates $\xi := x + ct$, $\eta := x - ct$ and write $v(\xi, \eta) = u(x,t)$ for solution in new coordinates.

chain rule: $\partial_x u = \partial_\xi v + \partial_\eta v$, $\partial_t u = c\partial_\xi v - c\partial_\eta v$

Again: $\partial_{xx} u = \partial_{\xi\xi} v + \partial_{\xi\eta} v + \partial_{\eta\xi} v + \partial_{\eta\eta} v$

$\partial_{tt} u = c^2 \partial_{\xi\xi} v - c^2 \partial_{\xi\eta} v - c^2 \partial_{\eta\xi} v + c^2 \partial_{\eta\eta} v$

u,v smooth: $\underbrace{\partial_{tt} u - c^2 \partial_{xx} u}_{0} = c^2(-4\partial_{\eta\xi} v)$

PDE in new coordinates: $\partial_{\eta\xi} v(\xi, \eta) = 0$

Integrate wrt η: $\partial_\xi v(\xi,\eta) = F(\xi)$ for some $F: \mathbb{R} \to \mathbb{R}$

Integrate wrt ξ: $v(\xi, \eta) = g(\eta) + f(\xi)$ for some $f, g: \mathbb{R} \to \mathbb{R}$ $f' = F$

General solution: $\boxed{u(x,t) = g(x-ct) + f(x+ct)}$

when does this work?

⑧ Initial value Problem (Cauchy Problem), D'Alembert's formula: Cauchy problem for wave equation in 1d on $\mathbb{R}$ is finding a solution $u: \mathbb{R} \times (0,\infty) \to \mathbb{R}$ s.t.

$$\partial_{tt} u(x,t) = c^2 \partial_{xx} u(x,t) \qquad (x,t) \in \mathbb{R} \times (0,\infty)$$

$$u(x,0) = \Phi(x) \leftarrow \text{initial position} \qquad x \in \mathbb{R}$$

$$\partial_t u(x,0) = V(x) \leftarrow \text{initial velocity} \qquad x \in \mathbb{R}$$

Now solving, we know $u(x,t) = g(x-ct) + f(x+ct)$ in general so...

$\Phi(x) = u(x,0) = g(x) + f(x) \Rightarrow f'(x) + g'(x) = \Phi'(x)$

$V(x) = \partial_t u(x,0) = cf'(x) - cg'(x) \Rightarrow f'(x) - g'(x) = \frac{1}{c} V(x)$

$\Big\}$ add & subtract these (*)

(*) yields $f'(x) = \frac{1}{2}\left(\Phi'(x) + \frac{1}{c}V(x)\right)$, $g'(x) = \frac{1}{2}\left(\Phi'(x) - \frac{1}{c}V(x)\right)$

integrating: $f(x) = \frac{1}{2}\Phi(x) + \frac{1}{2c}\int_0^x V(r)\,dr + C_f$, $g(x) = \frac{1}{2}\Phi(x) - \frac{1}{2c}\int_0^x V(r)\,dr + C_g$

Note: $f(x) + g(x) = \Phi(x) \Rightarrow C_f = C_g = 0$. Finally w/ rearrangement

d'Alembert's Formula

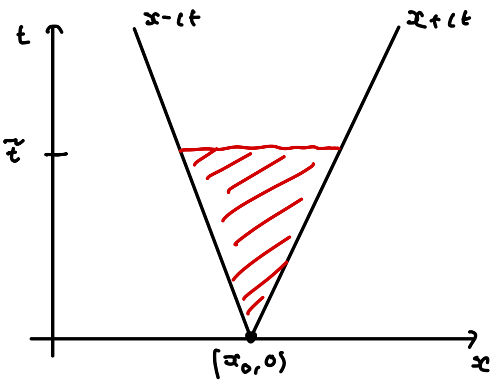
$$\boxed{u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\left(\Phi(x+ct) + \Phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(r)\,dr}$$

When do you have a smooth solution? $\Phi \in C^2(\mathbb{R})$, $V \in C^1(\mathbb{R})$

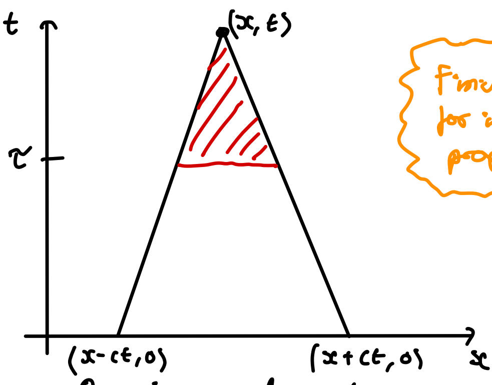
Pf: Set $v(x,t) = \Phi(x+ct)$, find $\partial_{tt} v$, $\partial_{xx} v$, and same for $v = \Phi(x-ct)$ so 1st term in solution vanishes.

Then use Leibnitz's rule for 2nd term & find $\frac{d^2}{dx^2}\int_{x-ct}^{x+ct} V(r)\,dr$, $\frac{d^2}{dt^2}\int_{x-ct}^{x+ct} V(r)\,dr$

(9) Principle of Causality: Information propagates w/ finite speed. Consider an earlier time $\tau \in [0, t)$



Domain of influence



Domain of dependence

Finite rate for information propagation

what values are later influenced by $u(x,t)$? for $\tilde{t} > t$, only $u(y, \tilde{t})$ with $y \in [x - c(\tilde{t} - t), x + c(\tilde{t} - t)]$

value of u at (x,t) depends only on value of $u(y,\tau)$ for $y \in [x - c(t-\tau), x + c(t-\tau)]$

The natural energy for wave eqn

(10) Preservation of energy: $E_k(t) = \underbrace{\int_{\mathbb{R}} \frac{1}{2}(\partial_t u)^2(x,t)\,dx}_{\text{kinetic}}$ $E_p(t) = \underbrace{\int_{\mathbb{R}} \frac{1}{2}c^2(\partial_x u)^2(x,t)\,dx}_{\text{potential}}$

Total $E_{WE}(t) = E_k(t) + E_p(t)$

wave equation

If (1) E_{WE} is diff wrt t, (2) $u(x,t)$ & partials satisfy $u(x,t) \to 0$ as $x \to \pm\infty$ $\forall t$ and (3) all p.d. of u upto order two are integrable wrt x over $\mathbb{R}$ $\forall t$ then total energy $E_{WE}(t)$ constant over time.

integrable by parts 2nd term

Pf: $E'_{WE}(t) = \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}c^2(\partial_x u)^2\,dx = \cdots = \int_{\mathbb{R}} \left(\partial_t u(\partial_{tt} u - c^2 \partial_{xx} u)\right)dx = 0$

take this in

$\cdots = \int_{\mathbb{R}} \partial_t u\, \partial_{tt} u\, dx + \int_{\mathbb{R}} c^2 \partial_x u\, \underbrace{\partial_x(\partial_t u)}_{\text{use this}}\, dx$ By parts this...

(11) <u>Uniqueness w/ an energy method:</u> Assume $\Phi \in C^2(\mathbb{R})$, $V \in C^1(\mathbb{R})$. Then $\exists$ unique solution $u \in C^2(\mathbb{R}\times(0,\infty))$ to IVP for 1d wave equation given by d'Lambert's formula.

<u>Pf:</u> Let $u^{(1)}, u^{(2)}$ be two solutions. Set $w = u^{(1)} - u^{(2)}$, then initial conditions cancel & w satisfies

$$\underbrace{\partial_{tt} w = c^2 \partial_{xx} w}_{(x,t)\in(\mathbb{R}\times(0,\infty))}, \quad \underbrace{w(x,0)=0, \quad \partial_t w(x,0) = 0}_{\forall x \in \mathbb{R}} \quad (*)$$

These are the hypothesis needed for energy preservation! So

$$E_{wE}(t) = \frac{1}{2}\int_{\mathbb{R}} (\partial_t w)^2(x,t) + c^2(\partial_x w)^2(x,t)\,dx \qquad \text{constant.}$$

But by $(*)$ $E_{wE}(t) = 0 \Rightarrow E_{wE}(t) = E_{wE}(0) = 0$

so $\partial_t w(x,t) = \partial_x w(x,t) = 0 \quad \forall x,t \quad \Rightarrow w$ constant. $w(x,0)=0 \Rightarrow u^{(1)} = u^{(2)}$ ∎

(12) <u>Initial boundary value problem & rescaling:</u> Now consider a finite spatial domain $x \in (0,L)$, $L>0$. PDE reads

Think vibrating spring...

$$\begin{aligned} \partial_{tt} u(x,t) &= c^2 \partial_{xx} u(x,t) && (x,t)\in(0,L)\times(0,\infty) \\ u(x,0) &= \Phi(x) && x \in (0,L) \\ \partial_t u(x,0) &= V(x) && x \in (0,L) \end{aligned} \quad (*)$$

Fix endpoints at 0

Dirichlet Boundary conditions

$$\left.\begin{aligned} u(0,t)&=0 \\ u(L,t)&=0 \end{aligned}\right\} t \in (0,\infty]$$

$\boxed{\text{Rescale the problem to get } c=1, \ L=\pi}$ How???

Set $\tilde{x} = \frac{x\pi}{L}$ so $\tilde{x}\in[0,\pi]$, set $\tilde{t} = \frac{t\pi c}{L}$ (so it works)

in practice set this as $\alpha \in \mathbb{R}$ & work out what is has to be based on x scaling

Set $\tilde{u}(\tilde{x},\tilde{t}) = u(x,t)$ so what does $\tilde{u}$ solve if u solves $(*)$?

$$\partial_t u = \frac{\partial}{\partial t}\tilde{u}(\tilde{x},\tilde{t}) = \tilde{u}_{\tilde{x}}\cdot\overbrace{\frac{d\tilde{x}}{dt}}^{0} + \tilde{u}_{\tilde{t}}\frac{d\tilde{t}}{dt} = \frac{\pi c}{L}\partial_{\tilde{t}}\tilde{u}$$

$$\partial_{tt} u = \frac{\pi c}{L}\Big(\partial_{\tilde{x}\tilde{t}}\tilde{u}\,\overbrace{\frac{d\tilde{x}}{dt}}^{0} + \partial_{\tilde{t}\tilde{t}}\tilde{u}\cdot\frac{d\tilde{t}}{dt}\Big) = \Big(\frac{\pi c}{L}\Big)^2\partial_{\tilde{t}\tilde{t}}\tilde{u}$$

$$\partial_x u = \frac{\partial}{\partial x}\tilde{u}(\tilde{x},\tilde{t}) = \tilde{u}_{\tilde{x}}\cdot\frac{d\tilde{x}}{dx} + \tilde{u}_{\tilde{t}}\cdot\overbrace{\frac{d\tilde{t}}{dx}}^{0} = \frac{\pi}{L}\partial_{\tilde{x}}\tilde{u}$$

$$\partial_{xx} u = \cdots = \Big(\frac{\pi}{L}\Big)^2\partial_{\tilde{x}\tilde{x}}\tilde{u} \Rightarrow c^2\partial_{xx}u = \Big(\frac{\pi c}{L}\Big)^2\partial_{\tilde{x}\tilde{x}}\tilde{u}$$

$$\partial_{tt} u = c^2\partial_{xx} u$$

$$\cancel{\Big(\frac{\pi c}{L}\Big)^2}\partial_{\tilde{t}\tilde{t}}\tilde{u} \qquad \cancel{\Big(\frac{\pi c}{L}\Big)^2}\partial_{\tilde{x}\tilde{x}}\tilde{u}$$

$$\Rightarrow \boxed{\partial_{\tilde{t}\tilde{t}}\tilde{u} = \partial_{\tilde{x}\tilde{x}}\tilde{u}}$$

$\tilde{u}$ solves wave eqn w/ $c=1$

so $(*)$ becomes

$$\begin{aligned} \partial_{\tilde{t}\tilde{t}}\tilde{u}(\tilde{x},\tilde{t}) &= \partial_{\tilde{x}\tilde{x}}\tilde{u}(\tilde{x},\tilde{t}) && (\tilde{x},\tilde{t})\in(0,\pi)\times(0,\infty) \\ \tilde{u}(\tilde{x},0) &= \Phi\big(\tilde{x}\tfrac{L}{\pi}\big) && \tilde{x}\in(0,\pi) \\ \partial_{\tilde{t}}\tilde{u}(\tilde{x},0) &= \frac{L}{\pi c}V\big(\tilde{x}\tfrac{L}{\pi}\big) && \tilde{x}\in(0,\pi) \end{aligned}$$

{WLOG can set $c=1$, $L=\pi$}

(13) <u>Boundary conditions:</u> You have $(*)$. want $u:(0,\pi \text{ or } L)\times(0,\infty)\to\mathbb{R}$ w/ also

endpoints fixed (Think a string)

<u>Dirichlet B.C.</u>
$u(0,t) = u(L,t) = 0$
$\forall t \in (0,\infty)$

<u>Neumann B.C.</u>
$\partial_x u(0,t) = \partial_x u(\pi,t) = 0$
$\forall t\in(0,\infty)$ $L=\pi$, $c=1$

String can move freely up & down

(14) <u>Separation of variables:</u> consider problem $(*)$ w/ dirichlet boundary conditions.
Assume solution takes form $u(x,t) = X(x)\,T(t)$ $\quad X:(0,\pi)\to\mathbb{R}$, $T:(0,\infty)\to\mathbb{R}$

$\Rightarrow \partial_{tt}u(x,t) = T''(t)X(x)$, $\quad \partial_{xx}u(x,t) = X''(x)T(t)$, so wave eqn reads

$$T''(t)X(x) = X''(x)T(t) \Rightarrow \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

depend on t & x only & equal so must be constant.

$$T''(t) - \lambda T(t) = 0 \quad \text{and} \quad X''(x) - \lambda X(x) = 0 \qquad \text{[eigenvalue problem]}$$

(15) <u>Discussion of eigenvalue problem accounting for B.C.:</u> $0 = u(0,t) = X(0)\,T(t)$ and $0 = u(\pi, t) = X(\pi)\,T(t) \Rightarrow X(0) = X(\pi) = 0$ are B.C. multiply by $-X$ & integrate $(0,\pi)$:

$$0 = \int_0^\pi -X''(x)X(x) + \lambda |X(x)|^2\,dx = \int_0^\pi |X'(x)|^2 + \lambda |X(x)|^2\,dx - X'(\pi)\underbrace{X(\pi)}_{0} + X'(0)\underbrace{X(0)}_{0}$$

$$= \int_0^\pi |X'(t)|^2 dx + \lambda \int_0^\pi |X(x)|^2\,dx$$

<u>Cases on λ:</u>

① $\lambda > 0 \Rightarrow \int_0^\pi x^2\,dx = 0 \Rightarrow X(x) = 0 \Rightarrow u = 0$ trivial

② $\lambda = 0 \Rightarrow \int_0^\pi |X'|^2 dx = 0 \Rightarrow X$ constant $X(0) = 0$ so $u = 0$ boring!

Assume $\lambda < 0$, write $\lambda = -\beta^2$

$$X''(x) + \beta^2 X(x) = 0, \quad T''(t) + \beta^2 T(t) = 0 \Rightarrow \left\{ \begin{aligned} X(x) &= C\cos(\beta x) + D\sin(\beta x) \\ T(t) &= A\cos(Bt) + B\sin(Bt) \end{aligned} \right\}$$

Boundary conditions: $X(0) = 0 \Rightarrow C = 0,\ D \neq 0$
$X(\pi) = 0 \Rightarrow \beta = j \in \mathbb{N}$ $\Big\} \Rightarrow X(x) = D\sin(jx) \quad j \in \mathbb{N}$

(16) <u>General Series Solution:</u> As $u_j = \dots$ can use linearity & see that

Dirichlet: $$\boxed{u(x,t) = \sum_{j\in\mathbb{N}} \big(A_j\cos(jt) + B_j\sin(jt)\big)\sin(jt)}$$

$u(x,0) = \sum_{j\in\mathbb{N}} A_j \sin(jx) = \Phi(x)$

$\partial_t u(x,0) = \sum_{j\in\mathbb{N}} B_j\, j \sin(jt) = \Psi(x)$

fourier series...

Neumann: $$\boxed{u(x,t) = \sum_{j\in\mathbb{N}^0} \big(A_j\cos(jt) + B_j\sin(jt)\big)\cos(jt)}$$

(17) <u>Uniqueness & Stability for IBVP:</u> Use energy method to establish stability & uniqueness.

Idea is to set $w = u^{(1)} - u^{(2)}$ then define its energy by

$$E_{wE}(t) = \int_0^L \tfrac{1}{2}\big(\partial_t w(x,t)\big)^2 + \tfrac{1}{2}c^2\big(\partial_x w(x,t)\big)^2\,dx$$

Then show $E'_{wE}(t) = 0$. Then $E_{wE}(t) = E_{wE}(0)$ so use BC.

<u>Chapter 4 - Fourier Series</u>

(18) <u>Definition of trigonometric polynomials / series:</u> $n \in \mathbb{N}$, trigonometric polynomials of degree $2n$ on $[-\pi,\pi]$ are

$$\sum_{k=-n}^{n} c_k e^{ikx} \qquad c_k \in \mathbb{C}, \quad x \in [-\pi,\pi]$$

If $n \to \infty$, these are Fourier series. use $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(19) <u>Relation between real / complex valued versions:</u>

$$a_0 + \sum_{k=1}^{n} \big(a_k\cos(kx) + b_k\sin(kx)\big) = \sum_{k=-n}^{n} c_k e^{ikx} \qquad c_k = \begin{cases} \tfrac{1}{2}(a_k - ib_k) & k > 0 \\ \tfrac{1}{2}(a_{-k} + ib_{-k}) & k < 0 \\ a_0 & k = 0 \end{cases}$$

<u>Note:</u> $c_{-k} = \overline{c_k}$ for $k \in \{-n, \dots, n\}$

Given $\phi : [-\pi,\pi] \to \mathbb{R}$ such as $\phi(x) = \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{ikx} \qquad \tilde{c}_k \in \mathbb{C},\ k \in \mathbb{Z}$

Then Fourier coefficients $\boxed{\hat{\phi}(k) = \tilde{c}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x) e^{-ikx}\, dx}$ $k \in \mathbb{Z}$

And we write $S_n(\phi)(x) = \sum_{k=-n}^{n} \hat{\phi}(k) e^{ikx}$ $n \in \mathbb{N} \cup \{\infty\}$

If ϕ is odd, $\hat{\phi}(-k) = -\hat{\phi}(-k)$ and fourier series is sin series

$$S_n(\phi)(x) = 2i \sum_{k=1}^{n} \hat{\phi}(k) \sin(kx)$$

If ϕ is even, $\hat{\phi}(-k) = \hat{\phi}(k)$ and fourier series is cos series

$$S_n(\phi)(x) = \hat{\phi}(0) + 2\sum_{k=1}^{n} \hat{\phi}(k)\cos(kx)$$

(20) Optimality of $\hat{\phi}(k)$ & formula: $\boxed{\hat{\phi}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x) e^{-ikx}\, dx}$

Note: $\underbrace{\int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\, dx = \begin{cases} 2\pi & k=l \\ 0 & k \neq l \end{cases}}_{\text{orthogonality property}}$

So the fourier coefficients minimize $\int_{-\pi}^{\pi} \left| \phi(x) - \sum_{k=-n}^{n} \hat{\phi}(k) e^{ikx} \right|^2 dx$

Pf: split $x^2 = x \cdot x$ w/ necessary complex conjugates.

(21) Useful inequalities:

Bessel's inequality: $2\pi \sum_{k\in\mathbb{Z}} |\hat{\phi}(k)|^2 \leq \int_{-\pi}^{\pi} |\phi(x)|^2 dx$

Parseval's equality: If $\lim_{n\to\infty} \int_{-\pi}^{\pi} |\phi(x) - S_n(\phi)(x)|^2 dx = 0$

Then $\int_{-\pi}^{\pi} |\phi(x)|^2 dx = 2\pi \sum_{k\in\mathbb{Z}} |\hat{\phi}(k)|^2$

Riemann-Lebesgue: $\lim_{k\to\pm\infty} \hat{\phi}(k) = \lim_{k\to\pm\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x) e^{-ikx}\, dx = 0$

$\phi \in C([-\pi,\pi], \mathbb{C})$

Then fourier coefficients converge to zero

Equivalently, $\lim_{k\to\pm\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\cos(-kx)\, dx = 0$

& $\lim_{k\to\pm\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\sin(-kx)\, dx = 0$

(22) Dirichlet kernel & pointwise convergence: we can do some algebra to see that

$S_n(\phi)(x) = \cdots = \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})(x-z))}{\sin(\frac{1}{2}(x-z))} \phi(z)\, dz$ so DEFINE

$k_n(\theta) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ik\theta} = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$ Dirichlet kernel 2π periodic

and $S_n(\phi)(x) = \int_{-\pi}^{\pi} k_n(x-z)\phi(z)\, dz$ so can say if $\phi \in C^1(\mathbb{R})$ periodic $\phi(x+2\pi m) = \phi(x)$ then

$S_n(\phi)(x) \longrightarrow \phi(x) \qquad \forall x \in [-\pi, \pi]$

Proof: use dirichlet kernel & then some substitutions & then consider $\phi(x) - S_n(\phi)(x) = \ldots$ & use Riemann Lebesgue lemma for 1 term & then get failure for $\theta = 0$ so use L-Hopital's rule to show function std continuous & apply R-L. again.

(23) Decay of coefficients & uniform convergence: Assume $\phi \in C^s(\mathbb{R})$, 2π periodic $s \in \mathbb{N}$, for $k \neq 0$, have

$$\boxed{|\hat{\phi}(k)| \leq C|k|^{-s}} \qquad \text{where } C = \sup_{x \in [-\pi,\pi]} |\phi^{(s)}(x)| = \|\phi^{(s)}\|_\infty$$

decay of coefficients

Proof: repeated integration by parts $|\hat{\phi}(k)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \phi(x) e^{-ikx}\,dx\right|$
Boundary terms disappear $\because$ periodicity.

Prop: Let $\phi \in C^2(\mathbb{T})$ 2π periodic, then $S_n(\phi) \Longrightarrow \phi$ as $n \to \infty$

Proof: note $\sum_{k\in\mathbb{Z}} |\hat{\phi}(k)e^{ikx}| \leq \sum_{k\in\mathbb{Z}} |\hat{\phi}(k)| < C \sum_{k\in\mathbb{Z}} \frac{1}{k^2} < \infty$

So $S_n(\phi)$ is absolutely convergent. To what value?

$$|\phi(x) - S_n(\phi)(x)| = \left|\sum_{k\in\mathbb{Z}} \hat{\phi}(k)e^{ikx} - \sum_{|k|\leq n} \hat{\phi}(k)e^{ikx}\right|$$

$$= \left|\sum_{|k|>n} \hat{\phi}(k)e^{ikx}\right|$$

$$\leq 2C \sum_{k=n+1}^{\infty} \frac{1}{k^2} \xrightarrow{n\to\infty} 0$$

So $\sup_{x\in[-\pi,\pi]} |\phi(x) - S_n(\phi)(x)| \xrightarrow{n\to\infty} 0$ [uniform convergence]

(24) Uniform convergence if $|c_j| \leq C|j|^{-r}$: $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ $\quad \phi_n(x) = \sum_{|k|\leq n} c_k e^{ikx}$

Proof: $\sup_{x\in I} |\phi_n(x) - \phi_m(x)| = \sup_{x\in I}\left|\sum_{n<|k|<m} c_k e^{ikx}\right| \leq \sum_{n<|k|<m} |c_k| \leq C \sum_{n<|k|<m} \frac{1}{k^r}$

As $r > 1$, $\{\phi_n\}_n$ cauchy $\Rightarrow$ convergent (to ϕ)

(25) Application to wave equation: If $\Phi \in C^4(\mathbb{T})$ $V \in C^3(\mathbb{T})$ then $u(x,t) = \sum_{k\in\mathbb{N}} (A_k \cos(kt) + B_k \sin(kt))\sin(kx)$ solves.

$$\begin{aligned}
\partial_{tt} u(x,t) &= \partial_{xx} u(x,t) && (x,t) \in (0,\pi)\times(0,\infty)\\
u(x,0) &= \Phi(x) && x \in (0,\pi)\\
\partial_t u(x,0) &= V(x) && x \in [0,\pi)\\
u(0,t) &= 0 && t \in (0,\infty)\\
u(\pi,t) &= 0 && t \in (0,\infty)
\end{aligned}$$

Pf: $\left.\begin{aligned}\Phi(x) = u(x,0) &= \sum_{k\in\mathbb{N}} A_k \sin(kx)\\ V(x) = \partial_t u(x,0) &= \sum_{k\in\mathbb{N}} B_k k \sin(kx)\end{aligned}\right]$ fourier series sin series $\Rightarrow$ $\begin{aligned}A_k &= 2i\,\hat{\Phi}(k)\\ B_k &= \frac{2i}{k}\hat{V}(k)\end{aligned}$ & use results...

Chapter 5 – The Heat Equation

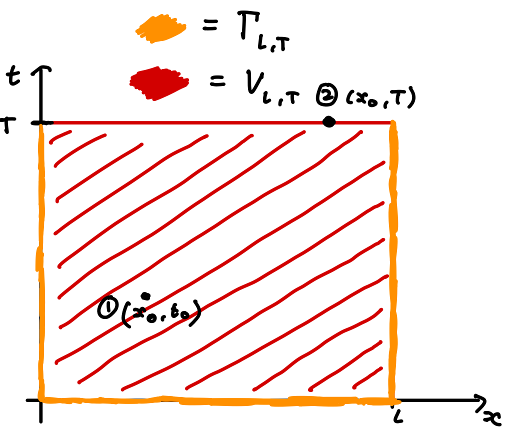
(26) **IBVP w/ homogeneous Dirichlet BC:** IBVP for heat equation in 1d w/ inhomogeneous Dirichlet BC is finding $u : (0,L) \times (0,\infty) \longrightarrow \mathbb{R}$ s.t.

$$\begin{aligned} \partial_t u(x,t) &= k \partial_{xx} u(x,t) && (x,t) \in (0,L)\times(0,\infty) \\ u(x,0) &= \Phi(x) && x \in [0,L] \\ u(0,t) &= g_0(t) && t \in [0,\infty) \\ u(L,t) &= g_L(t) && t \in [0,\infty) \end{aligned}$$

Note: ϕ, g_0, g_L satisfy compatibility conditions: $g_0(0) = \Phi(0)$, $g_L(0) = \Phi(L)$

(27) **Space time cylinder, parabolic boundary, weak maximum principle:** (is a strong but much harder to prove...) we define the space time rectangle (cylinder for $d > 1$) by

$$V_{L,T} = \{(x,t) \mid x \in (0,L),\ t \in (0,T]\} = (0,L) \times (0,T]$$



The parabolic boundary is

$$\Gamma_{L,T} = \{(x,t) \in \overline{V_{L,T}} \mid x \in \{0,L\} \text{ or } t = 0\}$$

(closure)

Thm (Maximum Principle): If $u \in C^2(\overline{V_{L,T}})$ solves the heat equation, then the maximum & minimum of u are attained on the parabolic boundary:

$$\max_{(x,t)\in\overline{V_{L,T}}} u(x,t) = \max_{(x,t)\in\Gamma_{L,T}} u(x,t)$$

$$\min_{(x,t)\in\overline{V_{L,T}}} u(x,t) = \min_{(x,t)\in\Gamma_{L,T}} u(x,t)$$

Who cares? Say $L = \pi$, $k = 1$, $g_0 = g_L = 0$, $\Phi(x) = \sin(2x)$.
Then $|u(x,t)| \le \max_{x \in \Gamma_{L,T}} |\Phi(x)| = 1$
without knowing solution, can show bounded by ± 1

(28) **Proof of maximum principal w/ peturbation argument:** Only do 1st assertion, 2nd $\boxed{\text{EXERCISE}}$
Define $M = \max_{(x,t)\in\Gamma_{L,T}} u(x,t)$. WTS $u(x,t) \le M$ $\forall (x,t) \in V_{L,T}$. Define for $\varepsilon > 0$

$$\boxed{v(x,t) = u(x,t) + \varepsilon x^2}$$

Can show diffusion inequality: $\partial_t v - k\partial_{xx} v = \overbrace{\partial_t u - k\partial_{xx} u}^{=0} - 2\varepsilon k < 0$ (✱)

Now we get two contradictions:

no interior maximum
① Suppose $(x_0,t_0) \in V_{L,T}$, $t_0 \neq T$ is a maximum of v. Then [A-level knowledge]
(derivative zero) (2nd derivative test)
$\partial_t v(x_0,t_0) = 0$, $\partial_{xx} v(x_0,t_0) \le 0 \Rightarrow \partial_t v(x,t) - k\partial_{xx} v(x,t) \ge 0$ ⨯ to (✱)

no boundary maximum
② Suppose (x_0,T), $x_0 \in (0,L)$ a maximum. Then $v(x_0,T) \ge v(x_0, T-\delta)$ so $\forall \delta \in [0,T]$

$$\partial_t v(x_0,T) = \lim_{\delta\to 0} \frac{v(x_0,T) - v(x_0,T-\delta)}{\delta} \ge 0$$

(2nd derivative test)
we still have $\partial_{xx} v(x_0,T) \le 0$ so $\partial_t v - k\partial_{xx} v \ge 0$ ⨯ to (✱)

But $\overline{V_{L,T}}$ is bbd & closed $\Rightarrow$ compact & v cts so necessarily has a maximum on $\Gamma_{L,T}$

$$\max_{(x,t)\in\overline{V_{L,T}}} v(x,t) = \max_{(x,t)\in\Gamma_{L,T}} v(x,t) = \max_{(x,t)\in\Gamma_{L,T}} (u(x,t) + \varepsilon x^2) \le M + \varepsilon L^2 \quad \text{so}$$

$$u(x,t) \le v(x,t) - \varepsilon x^2 \le \max_{\tilde{x},\tilde{t}\in\overline{V_{L,T}}} v(\tilde{x},\tilde{t}) - \varepsilon x^2 \le M + \varepsilon(L^2 - x^2) \quad \forall x,t \in V_{L,T}$$

ε arbitrary so $u(x,t) \le M$ $\forall (x,t) \in V_{L,T}$ ∎

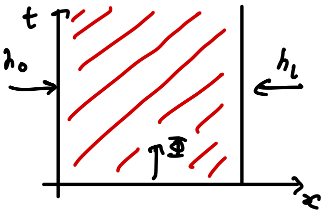
(29) Using maximum principal for uniqueness / stability: Let $u^{(i)} \in C^2([0,L]\times[0,T])$, $i=1,2$, be two solutions to (26), with initial data $\Phi^{(i)}$ & boundary data $g_0^{(i)}, g_L^{(i)}$. Then

$$\max_{(x,t)\in V_{L,T}} |u^{(1)} - u^{(2)}| \le \max\left\{ \max_{x\in[0,L]} |\Phi^{(1)}(x) - \Phi^{(2)}(x)|, \max_{t\in[0,T]} |g_0^{(1)}(t) - g_0^{(2)}(t)|, \max_{t\in[0,T]} |g_L^{(1)} - g_L^{(2)}| \right\}$$

If boundary data all agree, then solution unique.

Pf: just apply maximum principal then add absolute value signs. We know about the boundary so just bound it there on each side of the rectangle.

(30) IBVP w/ inhomogeneous Neumann conditions: The IBVP for the heat equation in 1d with inhomogeneous Neumann BC is finding $u: (0,L)\times(0,\infty) \longrightarrow \mathbb{R}$ s.t.



$$\begin{aligned} \partial_t u(x,t) &= k\,\partial_{xx} u(x,t) && (x,t)\in(0,L)\times(0,\infty) \leftarrow \text{behaviour inside} \\ u(x,0) &= \Phi(x) && x\in[0,L] \\ \partial_x u(0,t) &= h_0(t) && t\in[0,\infty) \\ \partial_x u(L,t) &= h_L(t) && t\in[0,\infty) \end{aligned}$$

with Φ, h_0, h_L smooth s.t. $h_0(0) = \Phi'(x)$, $h_L(0) = \Phi'(L)$

(31) Energy decay, Uniqueness & stability with Energy method: The natural energy for heat eqn is

$$\boxed{E_{HE,u}(t) = \int_0^L \frac{1}{2}u^2(x,t)\,dt}$$

To show uniqueness, $u^{(1)}, u^{(2)}$ solns

$$\left.\begin{aligned} \partial_t w &= k\,\partial_{xx} w \\ w(x,0) &= 0, \\ \partial_x w(0,t) &= 0 \\ \partial_x w(L,t) &= 0 \end{aligned}\right]$$ where $w = u^{(1)} - u^{(2)}$ and standard ranges.

Then calculate:

$$\frac{d}{dt}E_{HE,w}(t) = \int_0^L w\,\partial_t w\,dx = \int_0^L k\,w\,\underbrace{\partial_{xx} w}_{\text{integrate}}\,dx$$

differentiate

$\int u'v = [uv] - \int uv'$

$u = \partial_x u \qquad v = w$

$\frac{du}{\partial x} = \partial_{xx} u \qquad \frac{\partial v}{\partial x} = \partial_x u$

$$= -k\int_0^L \partial_x w\,\partial_x w\,dx + \underbrace{k\,w(L,t)\,\partial_x w(L,t)}_{=0} - \underbrace{k\,w(0,t)\,\partial_x (0,t)}_{=0}$$

$$= -k\int_0^L (\partial_x w)^2\,dx \le 0$$

$\boxed{\text{So } E_{HE,u}(t) \text{ only decreases in time}}$

bigger at start than end

Uniqueness: $w(x,0) = 0 \Rightarrow E_{HE,w}(t) = \int_0^L \frac{1}{2}w^2(x,t)\,dx \le E_{HE,w}(0) = 0$

so $u^{(1)} - u^{(2)} = w = 0$ $\forall x,t\in V_{L,T}$ $\Rightarrow$ unique

$\swarrow$ by $E_{HE,w} \ge 0$

$\Phi(x) = u(x,0)$

Stability: Different $\Phi^{(i)}$ $i=1,2$ boundary data. Still have energy decay so

$$\int_0^L \frac{1}{2}(u^{(1)} - u^{(2)})^2\,dx = E_{HE,w}(t) \le E_{HE,w}(0) = \int_0^L \frac{1}{2}(\Phi^{(1)} - \Phi^{(2)})^2\,dx$$

solution stable in mean squares sense wrt initial data.

(33) Superposition Principal: write $u = \underbrace{u^{(B)}}_{\text{boundary}} + \underbrace{u^{(I)}}_{\text{initial}} + \underbrace{u}_{\text{homogeneous}}$. Solve (26) with $u(0,t) = g_0(t)$, $u(L,t) = g_L(t)$

Set $h: [0,1] \to \mathbb{R}$ s.t. $h(0) = 0$, $h(1) = 1$.

Then $u^{(B)}(x,t) = g_0(t) + (g_L(t) - g_0(t))\,h(x)$ has $u^{(B)}(0,t) = g_0(t)$, $u^{(B)}(L,t) = g_L(t)$

Set $u^{(I)}(x,t) = \Phi(x) - u^{(B)}(x,0)$ & use compatability conditions

$\Rightarrow$ w satisfies $\partial_t w - k\,\partial_{xx} w = f(x,t)$ [f nasty in terms of g_0, g_L etc]

Need to understand this...

(34) Duhamel's Principal for inhomogeneous PDE: Now find $w: (0,L)\times(0,\infty) \longrightarrow \mathbb{R}$ s.t.

$$\partial_t w(x,t) = k\,\partial_{xx} w(x,t) + \underline{f(x,t)} \qquad (x,t)\in(0,L)\times(0,\infty)$$

$\leftarrow$ GIVEN

$w(x,0) = 0$, $x\in[0,L]$ $\qquad$ $w(0,t) = w(L,t) = 0$, $t\in[0,\infty)$ $\qquad$ (**)

HARD so displace by τ

Idea: Solve following problem family w/ parameter $\tau > 0$. — The start points in time

Find $v : (0,L) \times (\tau, \infty) \longrightarrow \mathbb{R}$ s.t.

$$(*) \quad \begin{cases} \partial_t v(x,t) = k\,\partial_{xx} v(x,t) & (x,t) \in (0,L)\times(\tau,\infty) \\ v(x,\tau) = f(x,\tau) & x \in [0,L] \\ v(0,t) = 0 & x \in [\tau,\infty) \\ v(L,t) = 0 & x \in [\tau,\infty) \end{cases}$$

Shifting along to a later start point

Good idea because...

Thm (Duhamel): Let $v(x,t,\tau)$ be a smooth solution to $(*)$ w/ param τ. Then

$$w(x,t) = \int_0^t v(x,t,\tau)\,d\tau \qquad (x,t) \in (0,L)\times(0,\infty)$$

solves our inhomogeneous problem $(**)$.

Pf: Substitute into $\partial_t w - k\,\partial_{xx} w = \dots = f(x,t)$

Example: Solve $\partial_t w - \partial_{xx} w = e^{-t}\sin(3x)$, $(x,t) \in (0,\pi)\times(0,\infty)$

$w(0,t) = w(\pi,t) = 0$, $w(x,0) = 0$, $L = \pi$, $k = 1$

STEP 1: Reformulate into displaced τ problem

STEP 2: Apply Duhamel.

initial condition

(35) **Separation of variables for homogeneous PDE, homogeneous BC, inhomogeneous IC:**

WLOG set $\tau = 0$, $\Phi(x) = f(x,\tau)$ & solve similarly to wave equation & get a series. Find $v : (0,L)\times(0,\infty) \longrightarrow \mathbb{R}$ s.t.

$$\begin{aligned} \partial_t v(x,t) &= k\,\partial_{xx} v(x,t) && (x,t) \in (0,L)\times(0,\infty) \\ v(x,0) &= \Phi(x) && x \in [0,L] \quad \text{inhomogeneous initial condition} \\ v(0,t) &= 0 && t \in [0,\infty) \\ v(L,t) &= 0 && t \in [0,\infty) \end{aligned} \Big\} \text{ homogeneous BC}$$

Separate: $v(x,t) = X(x)\,T(t)$, sub in, solve.
(exactly same as process for wave equation)

Fourier Series..

Get: $\Phi(x) = f(x,\tau) = v(x,0) = \sum_{j \in \mathbb{N}} D_j \sin(\beta_j x)$ [see notes for details...]

(36) **Cauchy Problem for the Heat Equation:** Only on real line w/ fewer initial conditions.

Cauchy or IVP to heat equation

$$\boxed{\begin{aligned} \partial_t u(x,t) &= k\,\partial_{xx} u(x,t) && (x,t) \in \mathbb{R}\times(0,\infty) \\ u(x,0) &= \Phi(x) && x \in \mathbb{R} \end{aligned}}$$

$\Phi \in C^1(\mathbb{R})$ & Φ has compact support

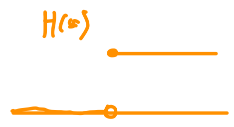
(37) **Delta Distribution & Heaviside functions:**

$$\boxed{\begin{aligned} &\delta : C^0(\mathbb{R}) \longrightarrow \mathbb{R} \\ &\delta(\psi) = \psi(0) \quad \forall \psi \in C^0(\mathbb{R}) \end{aligned}}$$

delta distribution

$$\boxed{H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}}$$

Heaviside step function



(39) **Derivation of the fundamental solution:** very long - doubt it'll come up...

Formula for differentiating an integral:

$$\frac{d}{dx}\int_a^x f(x,t)\,dt = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)\,dt$$

Really it's just Leibnitz's rule...

property	Heat Equation	wave Equation
Propogation speed	infinite. positive everywhere	finite - principal of causality [keeps info encoded in initial values & transports them]
Reversibility	Not well posed going backwards in time	Can invert time & still solves $u(x,t)$ & $u(x,-t)$ both solutions
maximum principle	holds	not true
Singularities	Smooths information	persists
Energy as $t \to \infty$	$u(x,t) \longrightarrow 0$ energy decreases.	Preserves total energy always non-zero. $(\partial_t u$ or $\partial_x u)$

chapter 6 - Laplace's Equation

(40) Classification of 2nd order PDEs:

PDE $\sum_{i,j=1}^{d} a_{i,j}(x)\, \partial_{x_i x_j} u(x) + \underline{\text{l.o.t}} = 0$ with symmetric coefficient matrix

lower order terms irrelevant for classification

$A(x) = (a_{i,j}(x))_{i,j=1}^{d}$ ->

elliptic = all $\pm$
hyperbolic = $+$ & $-$
parabolic = 0 & $\pm$

name	eigenvalues of $A(x)$
elliptic	d of same sign
hyperbolic	$d-1$ of same sign 1 of opposite
parabolic	one is zero, rest have same sign

wave equation

$\partial_{tt} u - \partial_{xx} u = 0$

$a_{i,j}(x,t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Heat equation

$\partial_t u - \partial_{xx} u = 0$

$a_{i,j}(x,t) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

When is this PDE eliptic / hyperbolic / parabolic?

$$x_2 \partial_{x_1 x_1} u + 4 \partial_{x_1 x_2} u + x_1 x_2 \partial_{x_2 x_2} u = 0$$

$$A = \begin{pmatrix} x_2 & 2 \\ 2 & x_1 x_2 \end{pmatrix} \Rightarrow \det(A) = x_1 x_2^2 - 4$$

(1) Parabolic if $\det(A) = 0$ [as one eigenvalue is then zero

(2) Eliptic if $\det(A) > 0$ so $x_1 x_2^2 > 4$

(3) Hyperbolic if $\det(A) < 0$ so $x_1 x_2^2 < 4$

[In 2×2 system $\det(A) = \lambda_1 \lambda_2$]

Identity:

$$\nabla \cdot (\nabla u\, G) = (\Delta u) G + \nabla u \cdot \nabla G \qquad \forall G \in C^2(\bar{\Omega})$$

Integrating Factor $\qquad I = e^{\int k}$ in

$$\frac{d}{dt}\phi(t) + k\,\phi(t) = 0$$

Leibnetz's rule

$$f(y) = \int_{a(y)}^{b(y)} g(y,z)\,dz$$

Then $\quad f'(y) = \underbrace{g(y, b(y))\,b'(y)}_{\text{boundary 1}} - \underbrace{g(y, a(y))\,a'(y)}_{\text{boundary 2}} - \int_{a(y)}^{b(y)} \partial_y g(y,z)\,dz$

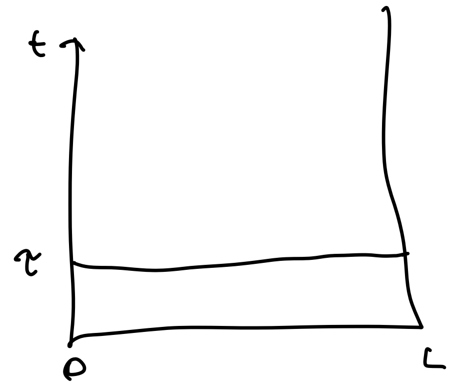
Boundary conditions

•

$$f(y) = \int_{a(y)}^{b(y)} f(y,z)\,dz$$

$$f'(y) = f(y, b(y))\,b'(y) - f(y, a(y))\,a'(y) + \int_{a(y)}^{b(y)} \partial_y f(y,z)\,dz$$

$$\boxed{\begin{aligned} \partial_t u - k\,\partial_{xx} u &= f(x,t) \\ u(x,0) &= 0 \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}}$$



Then $\quad u(x,t) = \int_0^t v(x,t,\tau)\,d\tau$

where $v(x,t)$ solves

$$\begin{aligned} \partial_t v - k\,\partial_{xx} v &= 0 && (x,t) \in (0,L) \times (\tau, \infty) \\ v(x,0) &= f(x,\tau) && x \in (0,L) \\ v(0,t) &= 0 && t \in (\tau, \infty) \\ v(L,t) &= 0 && t \in (\tau, \infty) \end{aligned}$$

$\xi = x + ct$

$\eta = x - ct$

$$\nabla \cdot (\nabla u\, G) = (\Delta u) G + \nabla u \cdot \nabla G$$

$\downarrow$

divergence thm

$$\int_{\partial\Omega} (\nabla u) G \cdot n\, ds \longleftarrow \text{zero.}$$