Algebra I Summary

16/4/23

Chapter 1 - MA106 review

① Basic definitions:

- A field k, is a set with addition & multiplication s.t. $(k,+)$ is abelian, $(k/\{0\}, \cdot)$ is abelian + distributivity.

- A vector space V over a field k is set V with vector addition & scalar multiplication.

- A basis of a vector space is a subset $B \subset V$ s.t. every $v \in V$ can be written uniquely as a finite linear combination of $b \in B$.

- A linear map $T: V \longrightarrow W$ "respects vector space structure" $[T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)]$ we can pick bases $E = \{e_1, \dots, e_n\}$, $F = \{f_1, \dots, f_m\}$ for V and W. Can write

$$T(e_j) = a_{1j} f_1 + \dots + a_{mj} f_m \qquad 1 \leq j \leq n$$

Then

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in k^{m \times n}$$

"matrix for T wrt chosen bases for V and W"

The columns of A are the images $T(e_1), \dots, T(e_n)$ of the basis vectors for V wrt basis $f_1, \dots, f_m$ of W.

- To change basis, consider this sequence of linear maps:

$$\underset{E'}{V} \xrightarrow{Id_V} \underset{E}{V} \xrightarrow{T} \underset{F}{W} \xrightarrow{Id_W} \underset{F'}{W}$$

Then as composition of linear maps corresponds to matrix multiplication

$$M(T)^{F'}_{E'} = M(Id_W)^{F'}_{F} \cdot M(T)^{F}_{E} \cdot M(Id_V)^{E}_{E'}$$

lower index: source
upper index: target

Chapter 2 - Jordan Canonical Form

$\boxed{\text{Aim: } J = P^{-1} A P}$

Find a basis for $T(v) = Av$ s.t. matrix is 'nice'

② Eigenvalues & Eigenvectors:

- Find some $0 \neq v \in V$, $\lambda \in k$ s.t. $T(v) = \lambda v$. λ eigenvalue for eigenvector v.

- $T: V \to V$. Matrix for T diagonal $\iff$ basis consisting of eigenvectors.

- $\lambda_1, \dots, \lambda_r$ distinct eigenvalues $\Rightarrow$ corresponding eigenvectors linearly indpt.

- $T: V \longrightarrow V$ $(\dim(V) = n)$ has n distinct eigenvalues. Then A is diagonalisable.

③ Minimal Polynomial:

Pick P by looking at kernels of $(A-\lambda I)^i$ and get eigenvectors for free!

- $A \in k^{n\times n}$, then $\exists p \in k[x]$, $\deg(p) \le n^2$ s.t. $p(A) = 0_{n\times n}$

Pf: $|\{I_n, A, \dots, A^{n^2}\}| = n^2+1$. $\dim(A^{n\times n}) = n^2 \Rightarrow$ linear dependency relation and set this as p.

- Def: The unique monic non-zero polynomial $\mu_A(x)$ of minimal degree with $\mu_A(A) = 0$ is the minimal polynomial of A.

Existence: $C_A(A) = 0 \Rightarrow \exists p \in k[x]$ of minimal degree [or use above]
Uniqueness: two monic $p_1(x)$, $p_2(x)$ s.t. $p_1(A) = p_2(A) = 0$. $p = p_1 - p_2$ contradicts minimality.

- $q \in k[x]$ s.t. $q(A) = 0$, then $\mu_A \mid q$.

Pf: write $q = s\mu + r$ s.t. $\deg(r) < \deg(\mu)$. If $r \neq 0$, then $r(A) = q(A) - s(A)\mu(A) = 0$ ⨳ to μ minimal $\Rightarrow r = 0 \Rightarrow \mu \mid q$.

④ Cayley Hamilton Theorem: $C_A(x)$ characteristic polynomial of $A \in k^{n\times n}$. Then $C_A(A) = 0_{n\times n}$ [not $0 \in k$!]

B_{ji} = delete jth row, ith column.
$c_{ji} = \det(B_{ji})$, then $\mathrm{Adj}(B)_{ij} = (-1)^{i+j} c_{ji}$

Pf: ① $B\,\mathrm{adj}(B) = \mathrm{adj}(B)\,B = \det(B) I_n$. Set $B = A - xI_n$.
So $\mathrm{Adj}(A - xI_n)(A - xI_n) = C_A(x) I_n$.

② $P(x) = \sum P_j x^j$, $Q(x) = \sum Q_k x^k$, $P_j, Q_k \in k^{n\times n}$ [matrices!]
Then $P(x)Q(x) = R(x) = \sum R_l x^l$ with $\sum_{j+k=l} P_j Q_k = R_l$
Note: $P(M)Q(M) = R(M)$ if $M \in k^{n\times n}$ commutes w/ all coefficients of Q.

③ Set $P(x) = \mathrm{adj}(A - xI_n)$, $Q(x) = A - xI_n$, $M = A$. $Q(A) = 0 \Rightarrow C_A(A) = 0$

⑤ Calculating minimal polynomial: just check the few possibilities given by Ⓐ $\mu_A \mid C_A$ Ⓑ λ eigenvalue of A. Then $\mu_A(\lambda) = 0$

Need $C_A(x)$ for this to work...

This always exists, horrible algorithm for large matrices in notes

Pf of Ⓑ: $Av = \lambda v \Rightarrow A^2 v = \lambda^2 v \Rightarrow \dots \Rightarrow A^n v = \lambda^n v \Rightarrow \forall p \in k[x]$ we have $p(A)v = p(\lambda)v \Rightarrow$ as $\mu_A(A)v = 0$ $\forall v \Rightarrow \mu_A(\lambda)v = 0$. $v \neq 0 \Rightarrow \mu_A(\lambda) = 0$

⑥ Jordan chains & Jordan blocks: $T: V \to V$ / $A \in \mathbb{C}^{n\times n}$

- $N_i(A,\lambda) = \{v \in V : (A - \lambda I_n)^i v = 0\}$ "generalised eigenspace of index i of A" — it's a kernel

- A jordan chain of length k is a sequence $v_1, \dots, v_k \in k^n$ s.t.

$$Av_1 = \lambda v_1, \quad Av_i = \lambda v_i + v_{i-1} \qquad 2 \le i \le k$$

- Jordan block of degree k with eigenvalue λ is $J_{\lambda,k}$ with

$$\gamma_{ij} = \begin{cases} \lambda & \text{if } i = j \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \qquad J_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

Matrix for $T|_W$ where W is subspace spanned by $\{v_1, \dots, v_k\}$

$T(v_1) = \lambda v_1$ ← 1st column
$T(v_2) = \lambda v_2 + v_1$ ← 2nd column

- Jordan basis for A is a basis of k^n w/ one or more jordan chains strung together. P is matrix w/ jordan basis as columns $\Rightarrow$ $\boxed{J = P^{-1}AP}$ need $k = \mathbb{C}$ to ensure at least one eigenvalue.
- $\boxed{\text{Main thm}}$: $A \in \mathbb{C}^{n \times n}$. Then $\exists$ jordan basis for A. Hence A is similar to a matrix J which is a direct sum of jordan blocks. The jordan blocks in J are uniquely determined by A.

Pf: very long... won't come up hopefully! Induction on $\dim(V) = n$.

⑦ <u>Consequences of JCF:</u> Let $A \in \mathbb{C}^{n \times n}$, $\{\lambda_1, \dots, \lambda_r\}$ eigenvalues of A.

Ⓐ $c_A(x) = (-1)^n \prod_{i=1}^{r} (x - \lambda_i)^{\alpha_i}$ Ⓑ $\mu_A(x) = \prod_{i=1}^{r} (x - \lambda_i)^{\beta_i}$

where α_i is the sum of the degrees of the jordan blocks of A for eigenvalue λ_i

where β_i is the largest among the degrees of the jordan blocks of A for eigenvalue λ_i

Ⓒ A is diagonalisable $\iff$ $\mu_A(x)$ has no repeated factors.

Pf: If $M = A \oplus B$, then $c_M(x) = c_A(x)\, c_B(x)$ and $\mu_M(x) = \text{lcm}\{\mu_A(x), \mu_B(x)\}$ and apply repeatedly.

Ⓓ Let λ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$. Let J be JCF of A. Then

(i) $\#\{\text{jordan blocks of } J \text{ w/ eigenvalue } \lambda\} = \text{nullity}(A - \lambda I_n)$

(ii) $\#\left\{\begin{matrix}\text{jordan blocks of } J \text{ w/} \\ \text{eigenvalue } \lambda \text{ of degree} \\ \underline{\text{at least}}\ i\end{matrix}\right\} = \text{nullity}((A - \lambda I_n)^i) - \text{nullity}((A - \lambda I_n)^{i-1})$ for $i > 0$

⑧ <u>Basic facts of the spectral theory of matrices:</u> (Grand Finale)

Ⓐ $A \in \mathbb{C}^{n \times n}$, $p \in \mathbb{C}[x]$, if λ is eigenvalue of A, $p(\lambda)$ eigenvalue of $p(A)$ (all take this form)

Pf: Assignment 1

Ⓑ $A \in \mathbb{C}^{n \times n}$, let $N_i(\lambda, A) = \ker((A - \lambda I_n)^i) = \{\text{generalised eigenvectors of } A \text{ wrt } \lambda \text{ of index } i\}$
$\Rightarrow$ every vector in $\mathbb{C}^n$ can be written as a sum of genuine/generalised eigenvectors.

Pf: $\exists P$ s.t. $J = P^{-1}AP$ and the P our our basis of generalised e.v.

Ⓒ $\exists S \in \mathbb{C}^{n \times n}$ s.t. $B = S^{-1}AS$
$\left(A, B \in \mathbb{C}^{n \times n} \text{ similar}\right) \iff \left|\begin{matrix}A \,\&\, B \text{ have same eigenvalues} \\ \lambda_1 = \mu_1, \dots, \lambda_k = \mu_k \\ \text{and } \dim N_i(A, \lambda_j) = \dim_i(B, \mu_j)\ \forall i,j\end{matrix}\right|$

Pf: '$\Rightarrow$' dimensions of corresponding generalised eigenspaces of similar matrices are the same $\because$ dimension = nullity$(A - \lambda I_n)^i$ dependent only on map T so invariant under basis change.
'$\Leftarrow$' Follows from uniqueness of JCF.

Chapter 3 – Functions of Matrices

(9) Lagrange Interpolation: speedy method for pen & paper $f(A)$

$\psi = M_A / C_A$ in practice...

- Suppose $\psi(M) = 0$. Divide by remainder: $z^n = q(z)\psi(z) + h(z)$ [This is division by remainder]

$$A^n = \cancel{q(A)\psi(A)} + h(A) = h(A)$$

Division w/ remainder for h hard;

Shortcut: roots of $\psi(z)$, say $\alpha_1, \dots, \alpha_k$, multiplicities $m_1, \dots, m_k$. Solve system of simultaneous equations in coefficients of $h(z)$:

$$f: z^n \qquad f^{(t)}(\alpha_j) = h^{(t)}(\alpha_j) \qquad \underbrace{1 \le j \le k}_{\text{roots}} \qquad \underbrace{1 \le t \le m_j}_{\text{multiplicities}}$$

Example: z^n

$M_A(z) = (z+2)^2$

$\Rightarrow h(z) = \alpha z + \beta$

$\therefore (-2)^n = -2\alpha + \beta$

$n(-2)^{n-1} = \alpha$

Example: e^{tz}

$C_A(z) = (1-z)(2-z)^2$

$\Rightarrow h(z) = \alpha z^2 + \beta z + \gamma$

So system is

$$\left\{\begin{aligned} e^t &= h(1) \\ e^{2t} &= h(2) \\ 2e^{2t} &= h'(2) \end{aligned}\right\}$$

multiple root

(10) Definition of a function of a matrix: Use z^n to generalise. Let

$J = P^{-1}AP$. $J = J_{\lambda_1, k_1} \oplus \dots \oplus J_{\lambda_t, k_t}$ is the JCF of A. We define

$$f(A) = P^{-1} f(J) P \quad \text{where} \quad f(J) = f(J_{\lambda_1, k_1}) \oplus \dots \oplus f(J_{\lambda_t, k_t})$$

where

$$f(J_{\lambda,k}) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \dots & \frac{f^{(k-2)}}{(k-2)!}(\lambda) & \frac{f^{(k-1)}}{(k-1)!}(\lambda) \\ 0 & f(\lambda) & \dots & \frac{f^{(k-3)}}{(k-3)!}(\lambda) & \frac{f^{(k-2)}}{(k-2)!}(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda) & f'(\lambda) \\ 0 & 0 & \dots & 0 & f(\lambda) \end{pmatrix}$$

Can use this or Lagrange interpolation to find e^A...

Lots of practice needed...

Chapter 4 – Bilinear Maps & Quadratic Forms

bilinear form $T: V \times V \longrightarrow K$ ($W = V$)

(11) **Definitions:** V, W are k-vector spaces. A bilinear map on V and W is a map $\tau: V \times W \longrightarrow k$ s.t.

(i) $T(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 T(v_1, w) + \alpha_2 T(v_2, w)$

(ii) $T(v, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 T(v, w_1) + \alpha_2 T(v, w_2)$

linear in both arguments

Pick basis $E: \{e_1, \ldots, e_n\}$, $F = \{f_1, \ldots, f_m\}$. Set $\alpha_{ij} = \tau(e_i, f_j)$

Pick coordinates for $v \in V$, $w \in W$. $v = x_1 e_1 + \cdots + x_n e_n$, $w = y_1 f_1 + \cdots + y_m f_m$

$$\tau(v, w) \overset{\text{by bilinearity}}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \tau(e_i, f_j) y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \alpha_{ij} y_j = \underline{v}^T A \underline{w}$$

(12) **Change of basis:** $P\underline{v}' = \underline{v}$, $Q\underline{w}' = \underline{w}$. Have $\underline{v}^T A \underline{w} = \tau(v, w) = (\underline{v}')^T A \underline{w}'$

so $\underline{v}^T B \underline{w} = (P\underline{v}')^T B (Q\underline{w}') = (\underline{v}')^T P^T A Q \underline{w}' \Rightarrow B = P^T A Q$

Note: Matrices A and B are congruent if $\exists$ invertible P s.t. $B = P^T A P$ [represent the same bilinear form in different bases].

Note: $\operatorname{rank}(\tau) := \operatorname{rank}(A) \Rightarrow$ rank well defined [independent of basis choice]

(13) **L/R radicals:** Since $\tau(v, w) = \underline{v}^T A \underline{w}$

$\ker(A) = \{v \in V : \tau(w, v) = 0 \quad \forall w \in V\}$ is the right radical of τ

$\ker(A^T) = \{v \in V : \tau(v, w) = 0 \quad \forall w \in V\}$ is the left radical of τ

$\operatorname{rank}(A) = \operatorname{rank}(A^T) \Rightarrow \dim\{\text{right/left radical}\} = \dim(V) - \operatorname{rank}(\tau) = n - r$

$\operatorname{rank}(\tau) = n \Rightarrow \tau$ is non-degenerate.

(14) **Symmetric / antisymmetric bilinear forms:** A bilinear form τ on V is

(i) symmetric if $\tau(v, w) = \tau(w, v) \quad \forall w, v \in V \Rightarrow A = A^T$

(ii) antisymmetric if $\tau(v, v) = 0 \quad \forall v \in V \Rightarrow A^T = -A$ + zeros on diagonal

Equivalently...

$T(v+w, v+w) = T(v, v) + T(v, w) + T(w, v) + T(w, w) = T(v, w) + T(w, v) = 0 \Rightarrow T(v, w) = -\tau(w, v)$

Prop: If $2 \neq 0$ in k, then any bilinear form $\tau = \underset{\text{symmetric}}{\tau_1} + \underset{\text{antisymmetric}}{\tau_2}$ [unique]

Pf: Put $\tau_1 = \underbrace{\tfrac{1}{2}(\tau(v, w) + \tau(w, v))}_{\text{symmetric}}$, $\tau_2 = \underbrace{\tfrac{1}{2}(\tau(v, w) - \tau(w, v))}_{\text{antisymmetric}}$

Uniqueness: if $\tau = \tau_1' + \tau_2'$, . . .

Hard?

(15) **Quadratic Forms:** V is a k-vector space. Then a quadratic form on V is a function $q: V \longrightarrow k$ s.t.

(i) $q(\lambda v) = \lambda^2 q(v) \quad \forall v \in V, \ \lambda \in k$

(ii) $\tau_q(v_1, v_2) = q(v_1+v_2) - q(v_1) - q(v_2)$ is a symmetric bilinear form on V.

Take a q
$\tau_q(v,w) = q(v+w) - q(v) - q(w)$
$q_{\tau_q}(v,v) = q(2v) - 2q(v)$
$= 4q(v) - 2q(v) = 2q(v)$

Note: there is a one-to-one association between quadratic forms & bilinear forms ($k: 0 \neq 2$)

$q \longmapsto \frac{1}{2}\tau_q, \quad \tau \longmapsto q$

(16) **Nice bases for quadratic forms:** V n-dim vector space w/ symmetric bilinear form τ (or q). Then

- $\exists$ basis $b_1, \dots, b_n$ of V & $\beta_1, \dots, \beta_n$ s.t.

$$\tau(b_i, b_j) = \begin{cases} \beta_i & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

- iff given any symmetric matrix A, $\exists$ invertible P s.t. $P^T A P$ diagonal.
- iff given any quadratic form q on V, $\exists$ basis $b_1, \dots, b_n$ & constants $\beta_1, \dots, \beta_n$ s.t.

$$q(x_1 b_1 + \dots + x_n b_n) = \beta_1 x_1^2 + \dots + \beta_n x_n^2$$

Pf: induction. start. pick v_1 s.t. $q(v_1, v_1) \neq 0$, $\beta_1 = q(v_1)$, then $\tau(w, v)$ ← kernel ← nullity $n-1$

Flowchart for finding this: Have quadratic form q. $B = (\beta_{ij})$ is matrix for q wrt arbitrary basis $b_1, \dots, b_n$. Modify:

Step 1: get $q(b_1) \neq 0$

case 1: $\beta_{11} \neq 0$, then done

case 2: $\beta_{11} = 0$ but $\beta_{ii} \neq 0$ then swap

case 3: $\beta_{ii} = 0 \ \forall i$, set $b_i := b_i + b_j$

note: be careful where change of basis coming from is

P = new basis vectors
$\Rightarrow$ will take from original matrix

Step 2: modify $b_2, \dots, b_n$ to make them orthogonal to b_1

$$\boxed{b_i = b_i - \frac{\beta_{1i}}{\beta_{11}} b_1}$$

why this?

$\tau(b_1, b_i - \frac{\beta_{1i}}{\beta_{11}} b_1) = \dots = 0$

Step 3: induct on n

(17) **(Sylvester's Theorem):** A quadratic form q over $\mathbb{R}$ has form

(t,u) is the signature of q

$$q(v) = \sum_{i=1}^{t} x_i^2 - \sum_{i=1}^{u} x_{t+i}^2$$

rank is the only invariant of a quadratic form over $\mathbb{C}$

wrt a suitable basis where $t + u = \text{rank}(q)$ Note: swap $\mathbb{R}$ to $\mathbb{C}$ & get $q(v) = \sum_{i=1} x_i^2$

(18) **(Sylvester's Law of Inertia):** q a quadratic form on V over $\mathbb{R}$. $e_1, \dots, e_n$ & $e_1', \dots, e_n'$ two bases for V s.t.

$$q(x_1 e_1 + \dots + x_n e_n) = \sum_{i=1}^{t} x_i^2 - \sum_{i=1}^{u} x_{i+t}^2, \quad q(x_1 e_1' + \dots + x_n e_n') = \sum_{i=1}^{t'} x_i^2 - \sum_{i=1}^{u'} x_{i+t'}^2$$

Then $t = t'$, $u = u'$ [signature invariant]

Pf: know $t + u = t' + u' = \text{rank}(q) \Rightarrow$ WTS $t = t' \Rightarrow$ assume $t > t'$

① $V_1 = \text{span}\{e_1, \dots, e_t\}$, $V_2 = \text{span}\{e_{t'+1}, \dots, e_n\}$. Then for $v, w \neq 0$
$\forall v \in V_1$, $q(v) > 0$, $\forall w \in V_2$ s.t. $q(w) < 0 \Rightarrow V_1 \cap V_2 = \{0\}$

② $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 + V_2) = t + (n - t') - \dim(V_1 \cap V_2) > 0$
$t - t' > 0$, $\dim(V_1 \cap V_2) \leq n$ ⨳

(19) Euclidean Spaces, orthonormal bases

→ signature is $(n,0)$, $t=n$

- The quadratic form q is positive definite if $q(v)>0 \quad \forall\, 0\neq v\in V$
- A vector space $V/\mathbb{R}$ w/ positive definite symmetric bilinear form τ is a Euclidean space.

wrt this τ

not standard dot product.. $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

orthogonal if not normalised.

- A basis for V s.t. $\tau(e_i,e_j) = \delta_{ij}$ is an orthonormal basis of V.

(20) Gram-Schmidt Process (orthonormalisation process):

Let V be a Euclidean space. $\dim V = n$. can modify basis $\{g_1,\dots,g_n\}$ to an orthonormal basis $\{f_1,\dots,f_n\}$ by:

Step 1: $f_1 := \dfrac{g_1}{|g_1|}$

inductively.

$$f'_{i+1} = g_{i+1} - \sum_{\alpha=1}^{i}(f_\alpha\cdot g_{i+1})\,f_\alpha$$

$$f_{i+1} = \frac{f'_{i+1}}{|f'_{i+1}|}$$

[basis change matrix $M(Id_V)^{(g_1,\dots,g_n)}_{(f_1,\dots,f_n)}$ is upper triangular!

(21) Orthogonal transformations:

- $T: V\longrightarrow V$ orthogonal if it preserves the scalar product: $T(v)\cdot T(w) = v\cdot w$

$T(v) = Av \Rightarrow T(v)\cdot T(w) = v^T A^T A w = v^T w \Rightarrow A^T A = I_n$

- $A\in\mathbb{R}^{n\times n}$ orthogonal if $A^TA = I_n$ $\quad$ $\det(A) = \pm 1$

(22) QR Decomposition:

Let $A\in\mathbb{R}^{n\times n}$, then can write $A = QR$ where Q is orthogonal and R is upper triangular.

Q: orthogonal basis $\quad$ R: change of basis matrix

Method: ① columns of A form a basis. Set this as $\{g_1,\dots,g_n\}$
② orthonormalise $\{g_1,\dots,g_n\} \longrightarrow \{f_1,\dots,f_n\}$ using Gram schmidt
③ $\{g_1,\dots,g_n\}$ is Q & find change of basis matrix R.

Useful! Solve $Ax=b$? Easy, $QRx = b \Rightarrow Rx = Q^Tb$ $\quad$ [R upper △ so easy, sub backwards!]

$$\boxed{R = \begin{pmatrix} f_1\cdot g_1 & f_1\cdot g_2 & f_1\cdot g_3 \\ 0 & f_2\cdot g_2 & f_2\cdot g_3 \\ 0 & 0 & f_3\cdot g_3 \end{pmatrix}}$$

(23) Nice orthonormal bases

- $T: V\longrightarrow V$ linear map on Euclidean space V. Then the linear map S s.t. $(Tv)\cdot w = v\cdot(Sw)$ $\forall v,w$, is the adjoint of T, T^*.

Note: ① Choose an orthonormal basis ⇒ matrix of T^* is transpose of T.
⇒ linear operator orthogonal iff $T^* = T^{-1}$. $\quad (Av)\cdot w = v\cdot(Bw)$

$v^TA^Tw = v^TBw \Rightarrow A^T = B$.

② T selfadjoint if $T^* = T$ [if bilinear form $T(v,w) = Tv\cdot w$ is symmetric.]

- $A\in\mathbb{R}^{n\times n}$ symmetric ⇒ all eigenvalues are real.

Pf: ① $\det(A-\lambda I_n) = 0$, $\mathbb{C}$ closed ⇒ at least one root $\lambda\in\mathbb{C}$. v eigenvector.
② $Av = \lambda v$, $A\bar v = \bar\lambda \bar v$ [as $A\in\mathbb{R}^{n\times n}$]
$\quad v^TA^T = \lambda v^T \rightarrow v^TA = \lambda v^T$ [as $A = A^T$]

$v^T\bar v = \alpha_1\bar\alpha_1 + \dots + \alpha_n\bar\alpha_n \in\mathbb{R}$

③ $\lambda v^T\bar v = v^TA^T\bar v = v^TA\bar v = \bar\lambda v^T\bar v \Rightarrow \lambda = \bar\lambda \Rightarrow \lambda\in\mathbb{R}$ (why?)

• Thm: Let V be a Euclidean space $\dim(V)=n$. Then

Ⓐ Given any quadratic form q on V, $\exists$ orthonormal basis $f_1,\dots,f_n$ of V & constants $\alpha_1,\dots,\alpha_n$ s.t.

$$q(x_1 f_1+\dots+x_n f_n) = \sum_{i=1}^{n} \alpha_i x_i^2$$

Ⓑ Given a self adjoint linear operator, $\exists f_1,\dots,f_n$ orthonormal basis $f_1,\dots,f_n$ of V consisting of eigenvectors of T.

Ⓒ $A\in\mathbb{R}^{n\times n}$ symmetric. $\exists P$ orthogonal s.t. P^TAP diagonal.

• (very useful!) $A\in\mathbb{R}^{n\times n}$ symmetric, λ_1,λ_2 distinct eigenvalues w/ v_1, v_2 eigenvectors. Then $v_1\cdot v_2=0$

If repeated eigenvalue, will need Gram Schmidt to orthonormalise.

Pf: $Av_1=\lambda_1 v_1$ & $Av_2=\lambda_2 v_2$ & consider $v_1^TAv_2$ . . .

(24) Quadratic Forms in Geometry: Can 'complete the square' to get general form of surfaces... lots of cases...

(25) Singular Value Decomposition: Given any $A\in\mathbb{R}^{k\times m}$, then there exist unique singular values $\gamma_1\geq\dots\geq\gamma_n>0$ and orthogonal matrices P and Q s.t.

$$\left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right) = P^TAQ \qquad [D=\operatorname{diag}(\gamma_1,\gamma_2,\dots,\gamma_n)]$$

The γ are the positive square roots of the non-zero eigenvalues of A^TA.

Note: If you have a symmetric matrix, SVD is just orthogonal diagonalisation with extra care needed for signs. Otherwise...

Step 1: eigenvalues of A^TA
$\gamma_i=\sqrt{\lambda_i}$

Step 2: want P,Q orthogonal s.t. $D=P^TAQ$
Find orthogonal basis of eigenvalues for A^TA
[This is Q]

Step 3: Find images of basis found & divide by corresponding γ_i

There is a complex story but non-exam.

Chapter 5 - Finitely Generated Abelian Groups

Bit of review of MA 136 - see that...

(26) New tools:

- The direct sum of groups $G_1, \dots, G_n$ is defined to be the set $\{(g_1, \dots, g_n) \mid g_i \in G_i\}$ (w/ componentwise addition).

- (1st isomorphism thm): Let $\phi : G \longrightarrow H$ be a homeomorphism w/ kernel k. Then $G/k \cong \mathrm{im}(\phi)$

- A finitely generated abelian group is called <u>free of rank n</u> if it is isomorphic to $\mathbb{Z}^n$.

- Elements $x_1, \dots, x_n$ of an abelian group G form an integral / $\mathbb{Z}$-basis of G $\Leftrightarrow$ they're linearly independent & span G.

- A matrix $P \in \mathbb{Z}^{n \times n}$ w/ $\det(P) = \pm 1$ is called uni-modular.

- Unimodular Smith normal form (SNF): $A \in \mathbb{Z}^{m \times n}$ rank$(A) = r$. can reduce A by a sequence of unimodular row & column operations to B. $B_{ii} = d_i$ for $1 \le i \le r$. $B_{ij} = 0$ otherwise. $d_i \mid d_{i+1}$ for $1 \le i \le r$

<u>Note</u>: gcd of all entries of A is d_1. [put d_1 in $(1,1)$ & continue]

(27) Results:

generators

- The group $\langle y_1, \dots, y_n \mid d_1 y_1, \dots, d_r y_n \rangle \cong \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \dots \oplus \mathbb{Z}/d_r \oplus \mathbb{Z}^{n-r}$

- (Fundamental thm of finitely generated abelian groups): If G is as described, then its isomorphic to a direct sum of cyclic groups. Precisely, if G is generated by n elements, then for some r $0 \le r \le n$, $\exists\, d_1, \dots, d_r \in \mathbb{Z}$ w/ $d_i > 0$, $d_i \mid d_{i+1}$ s.t.

  make matrix $\longrightarrow$ SNF to get the d_i

  $$G \cong \underbrace{\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \dots \oplus \mathbb{Z}/d_r}_{r \text{ finite cyclic groups}} \oplus \underbrace{\mathbb{Z}^{n-r}}_{n-r \text{ infinite cyclic groups}}$$

<u>Note</u>: $|G| = d_1 d_2 \dots d_r$ $\Rightarrow$ $n = 36$ order finite abelian $G \cong \mathbb{Z}/36$, $\mathbb{Z}/2 \oplus \mathbb{Z}/18$, $\mathbb{Z}/3 \oplus \mathbb{Z}/12$, $\mathbb{Z}/6 \oplus \mathbb{Z}/6$

- Let $G = G_1 \oplus \dots \oplus G_n$ be a finite abelian group. order$(g) = \mathrm{lcm}(g_1, \dots, g_n)$ [$g = (g_1, \dots, g_n)$] $\Rightarrow$ lcm of all components of G.
  [isomorphic groups have same # of elements $\boxed{\text{of each order}}$]

  $\mathbb{Z}/36$ has 1 element of order 36 $\Downarrow$ not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/18$

- Let H be the subgroup of $\mathbb{Z}^n$ generated by columns of $A \in \mathbb{Z}^{n \times n}$ [invertible in $\mathbb{Q}^{n \times n}$]. Index of H in $\mathbb{Z}^n = |\det(A)|$
  [order$(\mathbb{Z}^n / H) = |\det(A)|$]

  diagonal entries of SNF of A

<u>Pf</u>: Assignment 4: $\underset{=}{\alpha_1 \dots \alpha_r}$ $f_i(A) = \gcd(\{ |\det(S)| : S \text{ is an } i \times i \text{ submatrix of } A\})$

1. Put $A \xrightarrow{\text{SNF}} B$. $|\det(A)| = |\det(B)|$
2. $H \cong \mathbb{Z}/\alpha_1 \oplus \dots \oplus \mathbb{Z}/\alpha_r$

KEY CALCULATIONS

A quadratic form $q: V \longrightarrow \mathbb{R}$ is a map s.t.
$q(\lambda v) = \lambda^2 q(v)$
$\tau_q(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$ is a symmetric bilinear form

① Jordan Canonical Form

$J = P^{-1} A P$

STEP 1: Find J

- Find $c_A(x)$ and then $\mu_A(x)$ if matrix small. $\mu_A \mid c_A$ & $\mu_A(\lambda) = 0$
- Find nullity $(A - \lambda I)$ = # jordan blocks for this eigenvalue, & nullity $(A - \lambda I)^i$ − nullity $(A - \lambda I)^{i-1}$ = # jordan blocks of degree at least i if matrix is large.

STEP 2: Find P

DON'T CALCULATE EIGENVECTORS (you get them for free!) (unless chain length 1)

- You know how many vectors you need in each chain.
  $A v_k = \lambda v_k + v_{k-1} \Rightarrow (A - \lambda I) v_k = v_{k-1} \quad \dots \quad (A - \lambda I)^n v_n = 0$
  Pick vectors not in the kernel of the smaller powers of of $(A - \lambda I)^i$. $v_k \in \ker(A - \lambda I)^k$, $v_k \notin \ker(A - \lambda I)^{k-1}$

STEP 3: order P correctly

- will be wrong if you don't order following the chain.
  $T(v_1) = \lambda v_1$
  $T(v_2) = \lambda v_2 + v_1$
  $T(v_3) = \lambda v_3 + v_2$

  images are the columns & order matters!

$$\begin{pmatrix} \lambda & 1 & 0 & \\ 0 & \lambda & 1 & \\ \vdots & 0 & \lambda & \ddots \\ 0 & \vdots & \vdots & \\ & 0 & 0 & \end{pmatrix}$$

② Functions of Matrices

Choice 1: and $A = P^{-1} J P \Rightarrow f(A) = P^{-1} f(J) P$, $f(J) = f(J_{\lambda_1, k_1}) \oplus \dots \oplus f(J_{\lambda_n, k_n})$

$$f(J_{\lambda,k}) := \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2} f''(\lambda) & \frac{1}{6} f'''(\lambda) \\ 0 & f(\lambda) & f'(\lambda) & \frac{1}{2} f''(\lambda) \\ 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}$$

Spot the pattern!

Choice 2: (Lagrange Interpolation). Want $f(A)$. Divide by μ_A or c_A

$$\underbrace{f(x) = c_A(x) q(x) + h(x)}_{} \Rightarrow f(A) = \cancel{c_A(A) q(A)} + h(A)$$

division with remainder so if c_A is a ith degree polynomial then $\deg(h) = i - 1$

Then find $h(x)$ — λ are eigenvalues

$f(\lambda) = h(\lambda)$
$f'(\lambda) = h'(\lambda)$ for repeated roots [works $\because c_A(\lambda) = 0$]

③ Nice basis for quadratic forms

β is matrix for q

STEP 1: Have matrix for q wrt basis $\{b_1, \dots, b_n\}$ & modify:

- If $\beta_{11} \neq 0$ ✓, if $\beta_{11} = 0$, swap basis vectors / add basis vectors.
- orthogonalise, $b_i = b_i - \frac{\beta_{1i}}{\beta_{11}} b_1$ ← sets all basis vectors τ_q orthogonal to b_1

$$\tau\left(b_i - \frac{\beta_{1i}}{\beta_{11}} b_1, b_1\right) = \tau(b_i, b_1) - \frac{\beta_{1i}}{\beta_{11}} \tau(b_1, b_1) = 0$$

STEP 2: Find matrix wrt new basis: $B = P^T A P$ where A wrt start basis & P is modification from start basis.

④ Nice basis for maps on Euclidean spaces

Signature is $(n, 0)$
$q(v) > 0 \quad \forall v \in V$

Tools:

Ⓐ Gram Schmidt: Have a matrix / basis, take columns $g_1, \dots, g_n$.

- $f_1 = \dfrac{g_1}{|g_1|}$
- $f_i' = g_i - \sum_{\alpha=1}^{i-1} (f_\alpha \cdot g_\alpha) f_\alpha$
- $f_i = \dfrac{f_i'}{|f_i'|}$

the $\{f_1, \dots, f_n\}$ are now orthogonal

$$f_i \cdot f_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Ⓑ Eigenvectors: If you have a quadratic form with $q(v) > 0$ then $A^T = A$ and so if λ, μ distinct $\Rightarrow v_\lambda \cdot v_\mu = 0$
[if you have some repeated, then just run gram schmidt]

QR Decomposition:

STEP 1: $\{g_1, \dots, g_n\} \xrightarrow[\text{Gram-Schmidt}]{} \{f_1, \dots, f_n\}$

columns of $A \in \mathbb{R}^{n \times n}$ — columns of Q

STEP 2: Find R by tracking changes in terms of basis or

$$R = \begin{pmatrix} f_1 \cdot g_1 & f_1 \cdot g_2 & f_1 \cdot g_3 \\ 0 & f_2 \cdot g_2 & f_2 \cdot g_3 \\ 0 & 0 & f_3 \cdot g_3 \end{pmatrix}$$

⑤ Singular value Decomposition: $A \in \mathbb{R}^{n \times m}$. write $D = P^T A Q$

STEP 1: Find eigenvalues of $A^T A$. Set $\gamma_i = \sqrt{\lambda}$ & arrange as increasing.

STEP 2: Find orthogonal basis of eigenvectors for $A^T A$. This is Q.

STEP 3: $PD = AQ$ so find images of Q vectors under A & divide by corresponding γ_i to find P.

⑥ Smith Normal Form:

STEP 1: Find gcd & move to top left w/ INTEGER operations

STEP 2: Rinse & repeat till done.