Analysis III Summary

Chapter 1 – Riemann Integration

① **Thm 1.9:** $f:[a,b]\to\mathbb{R}$ bdd

f integrable $\iff$ $\forall \varepsilon>0 \;\exists P\in\mathcal{P}$ s.t. $U(f,P)-L(f,P)<\varepsilon$

Pf:  P_1 & P_2

R refine P_1 & P_2

② **Thm 1.14:** $f:[a,b]\to\mathbb{R}$ cts $\Rightarrow$ f uniformly cts

Pf: ① contradiction, not uniformly cts. Pick $\delta=\frac{1}{n}$
② $\{x_n\},\{y_n\}\subset[a,b]$, closed $\Rightarrow$ B-W & subsequences.
③ $x_{n_k}\to x$, $y_{n_k}\to y$ $\quad x=y \;\because\; |x-y_{n_k}|\le|x_{n_k}-x|+|x_{n_k}-y_{n_k}|<\frac{1}{n_k}\to 0$
④ $|f(x)-f(y)|=0>\varepsilon$ ※

③ **Thm 1.11:** $f:[a,b]\to\mathbb{R}$ cts $\Rightarrow$ f integrable

Pf: ① uniformly cts. Given ε, use δ: $|I_k|<\delta$ $\quad |f(x)-f(y)|<\frac{\varepsilon}{b-a}$
② cts so achieves max $M_k=f(x_k)$ and $m_k=f(y_k)$ on each I_k
③ sum over k

④ **Thm 1.15:** $f:[a,b]\to\mathbb{R}$ monotonic $\Rightarrow$ f integrable

Pf: ① Pick uniform partition $I_k=\left[a+\frac{b-a}{n}k,\; a+\frac{b-a}{n}(k+1)\right]$ $k\in\{0,\dots,n-1\}$
② M_k and m_k achieved at endpts.
③ Telescopic sum for $U(f,P)-L(f,P)$

⑤ **Thm 1.17:** $f:[a,b]\to\mathbb{R}$ integrable s.t. $f\le g$ $\Rightarrow$ $\int_a^b f\le\int_a^b g$

Pf: $g-f$ integrable, $g-f\ge 0$ $\Rightarrow$ $U(f-g,P)\ge 0$ $\Rightarrow$ $U(f-g)\ge 0$

⑥ **Coro 1.20:** $f:[a,b]\to\mathbb{R}$ cts $\Rightarrow$ $\exists c\in[a,b]$ s.t. $f(c)=\frac{1}{b-a}\int_a^b f$

Pf: ① $m(b-a)\le\int_a^b f\le M(b-a)$ $\Rightarrow$ $m\le\frac{1}{b-a}\int_a^b f\le M$

② f cts so by IVT, f obtains every value inbetween m & M

③ $\exists c\in[a,b]$ s.t. $f(c)=\cdots$

⑦ **Thm 1.26:** $f:[a,b]\to\mathbb{R}$ integrable, $\phi:\mathbb{R}\to\mathbb{R}$ cts $\Rightarrow$ $\phi\circ f$ is integrable

Pf: Hard & long...

change of variables formula & integration by parts

⑧ Thm 1.26: f, g integrable $\Rightarrow$ fg integrable

if also $\frac{1}{g}$ bdd $\Rightarrow$ $\frac{f}{g}$ integrable

Pf: ① $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ & use composition, $\phi(x) = x^2$

② g bdd $\Rightarrow$ g bdd away from 0. $\exists \varepsilon > 0$ s.t. $\varepsilon < |g|$ on $[a,b]$

$$\varphi(x) = \begin{cases} \frac{1}{x} & |x| > \varepsilon \\ \frac{1}{\varepsilon^2} x & |x| \le \varepsilon \end{cases}$$

$\varphi \circ g = \frac{1}{g}$ on $[a,b]$. g int, φ cts $\Rightarrow$ ✓

⑨ Thm 1.28: (FTC) $F: [a,b] \to \mathbb{R}$ cts on $[a,b]$, diff on (a,b). $F' = f$

Assume $f: [a,b] \to \mathbb{R}$ integrable. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Pf: ① Enough to take an arbitrary $P \in \mathcal{P}$ & show $L(f,P) \le F(b) - F(a) \le U(f,P)$

② Use MVT $\exists c_k \in I_k$ s.t. $(x_k - x_{k-1}) F'(c_k) = F(x_k) - F(x_{k-1})$

③ F cts so attains M_k and m_k.

④ Sum over k

⑩ Definition of improper integrals on $(a,b]$, $[a,b)$ and $[a,b] / \{c\}$

⑪ Thm 1.43: (Absolute comparison test). $f: [a,\infty)$ integrable on $[a,b]$ $\forall b > a$.

$$\int_a^\infty |f| < \infty \Rightarrow \int_a^\infty f \text{ converges.}$$ [absolutely convergent]

If $g: [a,\infty) \to [0,\infty)$ s.t. $|f| \le g$, $\int_a^\infty g < \infty \Rightarrow \int_a^\infty f$ absolutely convergent.

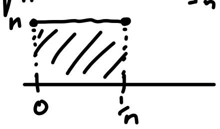
More counterexample functions "$\rightarrow$" pointwise "$\rightrightarrows$" uniformly

① $f_n = x$, $g_n = \frac{1}{n}$, $f_n g_n = \frac{x}{n}$ but $f_n g_n \not\rightrightarrows 0$ $\|f_n g_n - fg\|_\infty = \sup_{x \in \mathbb{R}} \left|\frac{x}{n}\right| = \frac{n}{n} = 1$

$f_n \rightrightarrows x$, $g_n \rightrightarrows 0$, $f_n g_n \to 0$

[2020 Exam]

② $f_n(x) = n \chi_{I_n}$ where $I_n = [0, \frac{1}{n}]$. Then $\int f_n = 1$ uniform convergence fails here.



But $\lim_{n \to \infty} f_n(x) = 0$ so $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \ne \int_a^b f(x)\,dx$ [2020 exam]

③ $f_n(x) = (x^2 + \frac{1}{n})^{\frac{1}{2}} \in C^1$ [$x^2 + \frac{1}{n}$ never vanishes] But $|x|$ not smooth at origin [2020 e]

$f_n \rightrightarrows |x|$ $(x^2 + \frac{1}{n})^{\frac{1}{2}} - |x| \le (x^2 + \frac{1}{\sqrt{n}})^{\frac{1}{2}} - |x| \le \frac{1}{\sqrt{n}}$ $\Rightarrow$ $|x|$ not differentiable.

[proves uniform convergence]

④ $f_n(x) = \frac{1}{n} \sin(n^2 x) \in C^\infty$ [infinitely smooth], but $f_n' = n\cos(n^2 x) \not\to 0$

$f_n \rightrightarrows 0$

f infinitely differentiable is not enough.

$f_n : \Omega \to \mathbb{R}$ $\quad \Omega$ fixed

Chapter 2 - Sequences & Series of Functions

$\boxed{\|f\|_\infty = \sup_{x\in\Omega} |f|}$

(12) **Def 2.1:** $f_n \longrightarrow f$ pointwise $\iff \forall x \in \Omega \quad \lim_{n\to\infty} f_n(x) = f(x)$

not ∞!

(13) **Def 2.8:** $f_n \rightrightarrows f$ uniformly $\iff \forall \varepsilon > 0 \ \exists N(\varepsilon)$ s.t. $\|f_n - f\|_\infty < \varepsilon \quad \forall n > N(\varepsilon)$

(14) **Thm 2.11:** (f_n) uniformly cauchy $\iff$ uniformly convergent.

Pf: '$\Rightarrow$' for each x, $f_n(x)$ cauchy in $\mathbb{R}$ so convergent. $\exists f$ s.t. $f_n(x) \longrightarrow f$ ptwise.

(15) **Thm 2.13:** (f_n) cts with $f_n \rightrightarrows f$, then f cts

Pf: WTS f cts at x_0. (A) f_n cts at $x_0 \Rightarrow \forall \varepsilon > 0 \ \exists \delta_\varepsilon$ s.t. $\forall x \in (x-\delta, x+\delta) \cap \Omega$
$|f_n(x) - f_n(x_0)| < \varepsilon/3$

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$
$$\leq \|f - f_n\|_\infty + |f_n(x) - f_n(x_0)| + \|f_n - f\|_\infty$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

(B) f_n uniformly convergent $\Rightarrow \forall \varepsilon > 0 \ \exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon) \ \|f_n - f\|_\infty < \frac{\varepsilon}{3}$

(16) **Thm 2.16:** $f_n : [a,b] \to \mathbb{R}$ integrable $f_n \rightrightarrows f$ uniformly, then f integrable $\blacksquare$

$$\left[\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n\right]$$

LONG!!!

Not easy!
v. rough
estimates
used - not clever...

Pf: Page long so could come up.

(1) Given $\varepsilon > 0$ find $P \in \mathcal{P}$ s.t. $U(f,P) - L(f,P) < \varepsilon$
  (a) $f_n \rightrightarrows f \Rightarrow \exists N$ s.t. $\|f_n - f\|_\infty < \frac{\varepsilon}{4(b-a)} \quad \forall n > N$
  (b) For n fixed, f_n integrable $\Rightarrow \exists P \in \mathcal{P}$ s.t. $U(f_n, P) - L(f_n, P) < \varepsilon/2$

(2) $\sup_{I_H} f \leq \|f_n - f\|_\infty + \sup_{I_H} f_n$, $\quad U(f, P_n) - L(f, P_n) \leq \cdots \leq \varepsilon$

(3) $\left| \int f_n - \int f \right| \leq \cdots \leq \|f_n - f\|_\infty (b-a) \longrightarrow 0$

Hard part is showing f is integrable. Once have limit f, easy to show $\lim \longleftrightarrow \int$ (3)

(17) **Thm 2.21:** f, $\frac{\partial f}{\partial t}$ cts on $[a,b] \times [c,d]$, then $\forall t \in (c,d)$

$$\frac{d}{dt} \int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

Pf: (1) set $F(t) = \int_a^b f(x,t)\,dx$, $\quad G(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$. WTS $F' = G$

(2) consider $\left| \frac{F(t+h) - F(t)}{h} - G(t) \right| \overset{(*)}{=} \left| \int_a^b \left(\frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t) \right) dx \right|$

(3) f cts diff on $[c,d] \Rightarrow$ MVT: $= \left| \int_a^b \frac{\partial f}{\partial t}(x,\tau) - \frac{\partial f}{\partial t}(x,t) \right| \leq \int_a^b \left| \frac{\partial f}{\partial t}(x,\tau) - \frac{\partial f}{\partial t}(x,t) \right| dx$

(4) Use cty of $\frac{\partial f}{\partial t}$ and take limits as $h \to 0$ in $(*)$.

(18) **Thm 2.22:** $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ cts $\Rightarrow \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$

Pf: Nasty proof...

Fails if not cts somewhere!!!

(19) **Thm 2.24:** $f_n \in C'[a,b]$ $\quad f_n \longrightarrow f$ pointwise, $f_n' \rightrightarrows g \Rightarrow f \in C'$, $f_n' \rightrightarrows f' = g$

Pf: $f_n' \rightrightarrows g$ so $\int_a^x g(y)dy = \int_a^x \lim_{n\to\infty} f_n' = \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} [f_n(x) - f_n(a)] = f(x) - f(a)$

integrable $\quad$ FTC $\quad$ limits

g cts $\Rightarrow f$ diff, $g' = f$

(20) **Thm 2.26:** $f_k : [a,b] \longrightarrow \mathbb{R}$ integrable. $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. $\Rightarrow \sum_{k=1}^{\infty} f_k$ int.

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$$

easy $\because$ previous thm

Pf: ① S_n finite sum $\Rightarrow$ integrable by additivity

② S_n converges uniformly $\Rightarrow$ S integrable & $\lim_{n\to\infty} \int S_n = \int \lim_{n\to\infty} S_n$

③ $\int S_n = \sum_{k=1}^{n} \int f_k$ and $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} f_k$

(21) **Thm 2.27:** $f_k : [a,b] \to \mathbb{R}$ $\quad f_k \in C'$ s.t. $S_n = \sum_{k=1}^{n} f_k$ converges ptwise. Assume $\sum_{k=1}^{n} f_k'$ uniformly converges.

Then

$$\left(\sum_{k=1}^{\infty} f_k\right)' = \sum_{k=1}^{\infty} f_k'$$

Pf: ① use $f_n \in C'[a,b]$, $f_n \longrightarrow f$ pointwise, if $f_n' \rightrightarrows g$ then $f_n' \rightrightarrows f' = g$

(22) **Thm 2.28:** (M-Test) $\quad f_k : \Omega \longrightarrow \mathbb{R}$. Assume for each k, $\exists M_k > 0$. s.t. $|f_k(x)| \le M_k$ $\forall x \in \Omega$

and $\sum_{k=1}^{\infty} M_k < \infty$ $\quad$ Then

$$\sum_{k=1}^{\infty} f_k$$

The actual inside summand

converges uniformly on Ω

Pf: Show $S_n = \sum_{k=1}^{n} f_k$ uniformly cauchy $\left[\sum_{k=1}^{\infty} M_k < \infty \Rightarrow \forall \varepsilon > 0 \; \exists N \text{ s.t. } \sum_{k=m+1}^{n} M_k < \varepsilon \;\; \forall n, m > N \right]$

$$|S_n(x) - S_m(x)| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \le \sum_{k=m+1}^{n} |f_k(x)| < \sum_{k=m+1}^{n} M_k < \varepsilon$$

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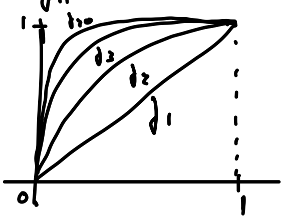
counter-example functions

(23) $f_n : [0,1] \longrightarrow \mathbb{R}$ $\qquad f_n(x) = x^{\frac{1}{n}}$

continuous



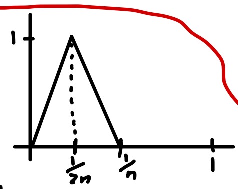
not continuous

$$\downarrow$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0,1] \end{cases}$$

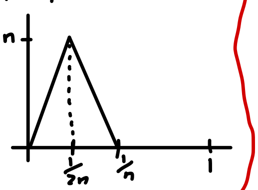
$$\underbrace{\lim_{n\to\infty} \lim_{x\to 0^+} f_n(x)}_{0} \neq \underbrace{\lim_{x\to 0^+} \lim_{n\to\infty} f_n(x)}_{1}$$

(24)
$$g_n(x) = \begin{cases} 2nx & x \in [0, \frac{1}{2n}] \\ -2n(x - \frac{1}{n}) & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$$



$g_n \xrightarrow{\text{pointwise}} 0$, $\sup_{x\in[0,1]} |g_n(x) - 0| = 1$

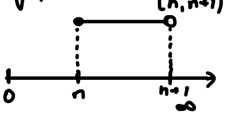
$$h_n(x) = \begin{cases} 2n^2 x & x \in [0, \frac{1}{2n}] \\ -2n^2(x - \frac{1}{n}) & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$$



$\sup_{x\in[0,1]} |h_n(x) - 0| \longrightarrow \infty$

Pointwise convergence non-uniform

(25) $f_n(x) = \chi_{[n,n+1)}(x)$ $\qquad \lim_{n\to\infty} \chi_{[n,n+1)}(x) = 0$



BUT:

$$1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} \chi_{[n,n+1)} \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} \chi_{[n,n+1)} = 0$$

(26) $g_n(x) = \frac{\sin(nx)}{n}$ $\qquad |g_n(x)| \le \frac{1}{n} \longrightarrow 0$ so $g_n \longrightarrow 0$

$g_n'(x) = \cos(nx)$ $\qquad \lim_{n\to\infty}(g_n'(x)) \neq \left(\lim_{n\to\infty} g_n(x)\right)'$

doesn't converge!

(27) Functions defined on $[0,1]$

$f_0(x) = \chi_{[0,1]}(x)$

$f_1(x) = \chi_{[0,\frac{1}{2}]}$ $\qquad f_2(x) = \chi_{[\frac{1}{2},1]}$

$f_3(x) = \chi_{[0,\frac{1}{4}]}$ $\quad f_4(x) = \chi_{[\frac{1}{4},\frac{1}{2}]}$ $\quad f_5(x) = \chi_{[\frac{1}{2},\frac{3}{4}]}$ $\quad f_6(x) = \chi_{[\frac{3}{4},1]}$

$\int f_n \longrightarrow 0$ but $f_n \not\longrightarrow 0$

Chapter 3 – Complex Analysis

$\boxed{h \in \mathbb{C}}$ $A: L(\mathbb{R}^n, \mathbb{R}^m)$ doesn't! Have a notion of division

exists

(28) **Def 3.5:** Ω open set, $z \in \Omega$. f complex differentiable at $z \iff \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$

Note: Derivative must exist for any path towards h. $h = \Delta x + i \Delta y$, $f(z) = u(z) + iv(z)$

$$\lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(z+h) - f(z)}{h} \Rightarrow \boxed{u_x = v_y \qquad u_y = -v_x}$$

(29) **Def 3.6:** ① $f: \Omega \to \mathbb{C}$ is analytic (holomorphic) at $z \in \Omega$ if $\exists$ neighbourhood $U \subset \Omega$ of z s.t. f is complex differentiable everywhere in U.

② f is analytic in Ω if it is analytic at every point of Ω.

③ f is entire if it is analytic in the whole of $\mathbb{C}$.

Note: Suppose $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists. Then f is cts at z. Pick a line of approach $h = \begin{pmatrix} \Delta x \\ 0 \end{pmatrix}$. Then $f'(z) = \dots = \partial_x f(z)$

all partial derivatives are continuous

(30) **Thm 3.7:** $f: \Omega \subset \mathbb{C} \to \mathbb{C}$, Ω open

f complex differentiable at $z = a + ib \in \Omega$ $\iff$ $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ has a differential at (a,b) that satisfies the Cauchy Riemann Equations

(31) **Thm 3.11:** (Ratio test) Consider $\sum_{n=0}^{\infty} a_n$. Assume $a_n \neq 0 \ \forall n$.

① If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ is convergent.

② If $\frac{|a_{n+1}|}{|a_n|} > 1 \quad \forall n > N \Rightarrow \sum_{n=0}^{\infty} a_n$ is divergent.

if $\lim \frac{|a_{n+1}|}{|a_n|}$ exists then $\sum_{n=0}^{\infty} a_n z^n$ has $R = \lim \frac{|a_n|}{|a_{n+1}|}$

[2020 Exam]

(32) **Thm 3.12:** (Root test) Consider $\sum_{n=0}^{\infty} a_n$. Then

① If $\limsup |a_n|^{1/n} < 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ converges

② If $\limsup |a_n|^{1/n} > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

(33) **Thm 3.33:** Given $(a_n)_{n=0}^{\infty}$, $\exists R \in [0, \infty]$ s.t.

$$\sum_{n=0}^{\infty} a_n z^n$$

if $\lim \frac{|a_{n+1}|}{|a_n|}$ exists

converges $\forall |z| < R$. diverges $\forall |z| > R$. $R = \frac{1}{\limsup |a_n|^{1/n}} \left(= \lim \frac{|a_n|}{|a_{n+1}|} \right)$

(34) **Thm 3.34:** Assume $\sum_{n=0}^{\infty} a_n z^n$ has RoC R. Then for $|z| < R$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is diff w/

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

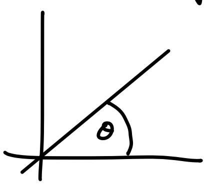
$$e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

$$\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

$$\cosh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n},$$

$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

$$\sinh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}.$$



$z = |z| e^{i\theta}$

multivalued

$\arg(z) = \{ \theta \in \mathbb{R} : z = |z| e^{i\theta} \}$

$\operatorname{Arg}(z) \in (-\pi, \pi]$ ← principle value

$\log(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z) + 2k\pi i$, $k \in \mathbb{Z}$

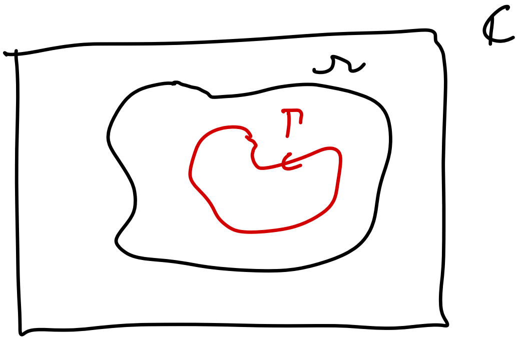
$\log(-1) = \operatorname{Log}|-1| + i \operatorname{Arg}(-1) + 2k\pi i = \pi i + 2k\pi i$

$\operatorname{Log}(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z)$

$z^\alpha = e^{\alpha \log(z)} = e^{\alpha \operatorname{Log}(z)} e^{2k\alpha\pi i} \Rightarrow 1^{1/q} = e^{2\pi i k/q}$, $k = 0, \dots, q-1$

Complex integration:

Some facts:



- $f : [a,b] \to \mathbb{C}$

$$\left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$$

- $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ along path $\Gamma \subset \Omega \subset \mathbb{C}$ parametrized by $\gamma : [a,b] \to \mathbb{C}$

$$\int_\Gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \_\_ + i\,\_\_$$

- $$\int_\gamma |dz| := \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = l(\gamma)$$ length of the curve

[Useful estimates]

- $f : \mathbb{C} \to \mathbb{C}$

$$\left| \int_\gamma f\,dz \right| \le \int_\gamma |f|\,|dz| := \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \le \max_{z\in\Gamma} |f(z)| \int_\gamma |dz| = \max_{z\in\Gamma} |f(z)|\, l(\gamma)$$

- $$\int_\gamma f\,d\bar{z} := \int_a^b f(\gamma(t))\,\bar{\gamma}'(t)\,dt$$

35 Thm 3.28: (Complex FTC) $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ analytic (Ω open) $f(z) = \frac{dF}{dz}$. f cts. $\gamma : [a,b] \to \mathbb{C}$ a C' curve. Then

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a))$$

Pf: chain rule

36 Thm 3.29 (Cauchy's thm) $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ analytic. Ω open simply connected domain. $\gamma \subset \Omega$ is C'.

$$\int_\gamma f(z)\,dz = 0$$

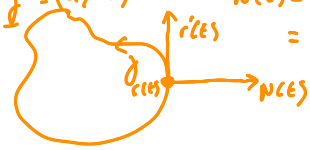
$f = u + iv$

$\bar{f} = (u, -v)$

$r(t) = (x(t), y(t))$

$N(t) = r'(t)^{\perp}$

$= (y'(t), -x'(t))$



Pf: know $\int_\gamma f\,dz = \text{circulation}(\underline{\bar f}) + i\,\text{flux}(\underline{\bar f})$

$$= \iint_\Omega \text{curl}(\underline{\bar f})\,dx\,dy + i \iint_\Omega \text{div}(\underline{\bar f})\,dx\,dy \qquad \text{[Green \& Gauss' Thms]}$$

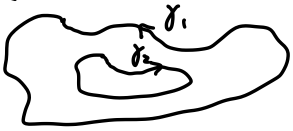
$$= \iint_\Omega \nabla \times \underline{\bar f}\,dx\,dy + i \iint_\Omega \nabla \cdot \underline{\bar f}\,dx\,dy$$

Note: $\nabla \times \bar{f} = \ldots$

Pf won't come up?

(37) **Thm 3.31:** $\Omega \subset \mathbb{C}$ region bdd by the simple curves γ_1 [exterior], γ_2 [interior]. positively oriented & f analytic on $\Omega \cup \gamma_1 \cup \gamma_2$. Then

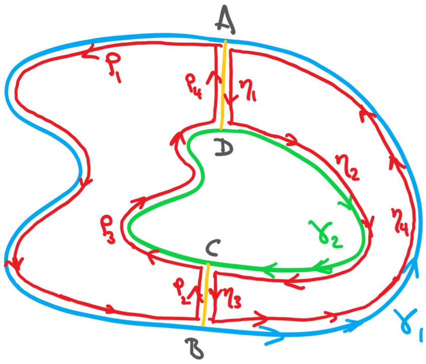
$$\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz = 0$$



Pf: Cauchy's Thm:

$$\int_\rho f\,dz = \int_{\rho_1} f\,dz + \int_{\rho_2} f\,dz + \int_{\rho_3} f\,dz + \int_{\rho_4} f\,dz$$

$$\int_\eta f\,dz = \int_{\eta_1} f\,dz + \int_{\eta_2} f\,dz + \int_{\eta_3} f\,dz + \int_{\eta_4} f\,dz$$



same paths in opposite directions cancel, add the equations & recover γ_1 & γ_2

(38) (Fundamental Contour Integral)

$$\int_{\partial B_r(a)} (z-a)^n\,dz = \begin{cases} 2\pi i & n=-1 \\ 0 & n \neq -1 \end{cases}$$

Pf: $\gamma(t) = a + re^{it}$ $t \in [0, 2\pi]$ & do the calculations

Restated

$$\int_{\partial B_r(z)} \frac{1}{w-z}\,dw = 2\pi i \quad (*)$$

Simple closed C' curve γ

(39) **Thm 3.33:** let $\gamma : [a,b] \to \mathbb{C}$ positively oriented simple closed C' curve. Assume f analytic on $I(\gamma) \cup \gamma$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dw \qquad \forall z \in I(\gamma)$$

Pf: Fix $z \in I(\gamma)$. Pick r s.t. $B_r(z) \subset I(\gamma)$. Then contour deform:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(w)}{w-z}\,dw$$

problem pt at $z=w$ so contour around there.

$$= \underbrace{\frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(z)}{w-z}\,dw}_{f(z) \text{ by } (*)} + \underbrace{\frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(w)-f(z)}{w-z}\,dw}_{E(z)}$$

f analytic in $I(\gamma)$ so given $\varepsilon > 0$ $\exists r > 0$ s.t.

$|f(w) - f(z)| < \varepsilon$ $\forall w \in B_r(z)$

Parametrize $\partial B_r(z)$ by $\gamma(t) = z + re^{it}$ for $t \in [0, 2\pi)$

$$|E(z)| = \left| \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(w)-f(z)}{w-z}\,dw \right| \leq \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{it}) - f(z)}{\cancel{re^{it}}}\, i\cancel{re^{it}}\,dt \right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \left| f(z+re^{it}) - f(z) \right| dt < \varepsilon$$

[arbitrary so $E(z) = 0$] ∎

(40) **Thm 3.35:** $\gamma : [a,b] \to \mathbb{C}$ positively oriented C' curve. Assume f analytic on $\overline{I(\gamma)}$. Then $f^{(n)}$ exists $\forall n \in \mathbb{N}$ &

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw \qquad \forall z \in I(\gamma)$$

Pf: Too long but same idea as above.

(41) Thm 3.36: (Taylor series expansion) let f be an analytic function on $B_R(a)$, $a \in \mathbb{C}$, $R>0$. Then $\exists\, c_n \in \mathbb{C}$ unique. $n \in \mathbb{N}$ s.t.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad \forall z \in B_R(a)$$

with the c_n given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw = \frac{f^{(n)}(a)}{n!}$$

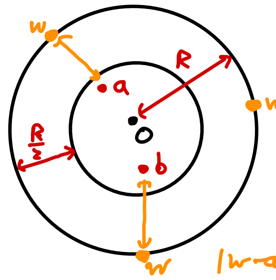
where γ is any p.o.s.c.c. in $B_R(a)$ with $a \in I(\gamma)$

Pf: ① choose γ $B_r(a)$ carefully s.t. $\frac{|z-a|}{|w-a|} < 1$

② $\frac{1}{w-z} = \frac{1}{w-a} \times \frac{w-a}{w-z} =$

③ see notes...

(42) Thm 3.38: (Liouville's thm). Let $f: \mathbb{C} \to \mathbb{C}$ be an analytic bounded function. Then f is constant.



$|w-a| < \frac{2}{R}$

$|w-b| < \frac{2}{R}$

$|w-a||w-b| < \frac{4}{R^2}$

Pf: Pick $a \neq b$ in $\mathbb{C}$. Pick R s.t. $2\max\{|a|,|b|\} < R$

Then for $w \in \partial B_R(0)$

will help bound integral

$$|w-a| > \frac{R}{2} \quad \text{and} \quad |w-b| > \frac{R}{2}$$

Use cauchy's formula

$$f(a) - f(b) = \dots = \frac{a-b}{2\pi i} \int_{\partial B_R(0)} \frac{f(w)}{(w-a)(w-b)}\,dw \Rightarrow |f(a)-f(b)| \leq \frac{|a-b|}{2\pi} \frac{M}{R^2/4} \cdot 2\pi R \longrightarrow 0$$

(43) Thm 3.39: (Fund. thm algebra). Every non-constant polynomial p on $\mathbb{C}$ has a root. ($\exists\, a \in \mathbb{C}$ s.t. $p(a) = 0$)

Pf: Contradiction: Assume $|p(z)| \neq 0$ $\forall z \in \mathbb{C}$.

① Define $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = \frac{1}{p(z)}$ [f is entire $\because$ composition of two holomorphic]

② As $|z| \to \infty$ $p(z) = \sum_{n=0}^{n} c_k z^k \longrightarrow c_n z^n \Rightarrow |p(z)| \longrightarrow \infty$ as $|z| \to \infty$

$\Rightarrow |p(z)| > 1$ $\forall |z| > R$

③ $f(z) = \frac{1}{p(z)}$ bounded in $\mathbb{C}$

ⓐ $|f| < 1$ $\forall |z| > R$

ⓑ bbd on $|z| \leq R$ [compact set] as f is cts.

④ Louiville's $\Rightarrow$ constant $\times$

(44) Thm 3.43: (Residue thm) $\gamma \subset \mathbb{C}$ contour, f analytic in $\overline{I(\gamma)} \setminus \{z_1, \dots, z_n\}$, $(z_k)_{k=1}^n \subset I(\gamma)$ f has poles at $z_1, \dots, z_n$. Then

$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z_k)$$

analytic

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \dots + \frac{c_{-1}}{z-a} + \phi(z)$$

$$\boxed{\operatorname{Res} f(a) = c_{-1} = \lim_{z \to a} (z-a) f(z)}$$

[for a simple pole]

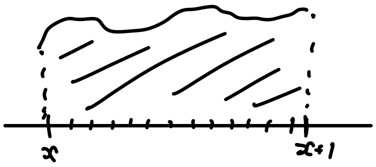
Pf:

You missed a crucial lemma that would've helped you in exam!

Sneaky exam tricks:

① $f_n : [0,2] \to \mathbb{R}$ $\qquad$ $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \frac{1}{n} \underbrace{\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)}$$



$$f_n(x) \longrightarrow \int_x^{x+1} f(y)\,dy$$

$|I_k| = \frac{1}{n}$

so close to an integral appear!

$$\|f_n - f\|_\infty = \sup_{x \in [0,1]} \left| \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) - \int_x^{x+1} f(t)\,dt \right|$$

$$= \sup_{x \in [0,1]} \left| \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f\left(x + \frac{k}{n}\right) dt - \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\,dt \right|$$

$$= \sup_{x \in [0,1]} \left| \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f\left(x + \frac{k}{n}\right) - f(t)\,dt \right|$$

$$\leq \sup_{x \in [0,1]} \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} \underbrace{\left| f\left(x + \frac{k}{n}\right) - f(t) \right|}_{\leq \varepsilon} dt$$

as cts on bdd interval $\Downarrow$ uniformly cts.

$\longrightarrow 0$ as ε arbitrary

Using cauchy reimann: get all in terms of u or u_x, v_x not x so can compare.