Asymptotics

Big O notation

$f(x) = O(g(x))$ as $x \to a$ $\iff$ $\left|\frac{f(x)}{g(x)}\right|$ bdd as $x \to a$

little o notation

$f(x) = o(g(x))$ as $x \to a$ $\iff$ $\left|\frac{f(x)}{g(x)}\right| \to 0$ as $x \to a$

Capital Θ

$f(x) = \Theta(g(x))$ as $x \to a$ $\iff$ $f = O(g)$ & $g = O(f)$

Asymptotic definitions

Series: $f(x) \sim \sum_{j=1}^{N} f_j(x)$ as $x \to a$

$\forall$ finite $m \in \mathbb{N}$, $f(x) - \sum_{j=1}^{m} f_j(x) = o(f_m(x))$

$[\varepsilon^3 = o(\varepsilon^2)]$

Sequence: $(\alpha_n)_{n\geq 1}$ s.t. $\alpha_{n+1}(x) = o(\alpha_n(x))$ as $x \to a$

Given a sequence $(\alpha_n)_{n\geq1}$, the asymptotic series

$f \sim \sum_{j=1}^{N} f_j \alpha_j(x)$, then f_j coeff. unique

$f(x, \varepsilon) \sim \sum_{j=1}^{N} f_j(x) a_j(\varepsilon)$ as $\varepsilon \to 0$ is a poincaré expansion

Poincare expansions have unique coefficients f_j, given $(\alpha_n)_{n\geq 1}$

Singular Expansions

(all pretty general, particular to integrals)

- Rescale 1st. (include terms left out when set $\varepsilon = 0$)
- For non-linear problems, rescale both f & x
- For some problems, need different rescalings in different places.
  - use correct scaling in correct place & match

Matching

Matching

Matching M

If we have different expansions in different places, need to match:

(a) Ad hoc: look at it by eye (few checks could be wrong)

(b) Van Dykes Matching Rule (Sheet 4 Q5)

$E_p H_Q f \overset{?}{=} H_Q E_p f$

(c) intermediate variables (did this w/ integrals (Sheet 2 Q2,3), but non-exam for ODEs).

Composite expansions: $C = E_p f + H_Q f - E_p H_Q f$

(not poincare $\Rightarrow$ not unique, Sheet 4 Q5)

Regular Expansions

① Iteration method

② Successive approximation
- Set $\varepsilon = 0$, solve for f_0
- Set $f = f_1 + e_1$, found exact eqn for e_1
- Approx solve for e_1 (by setting $\varepsilon = 0$), to get $\hat{e}_1$. $f = f_0 + \hat{e}_1 + e_2, \ldots$

③ Series Expansion
- Guess series

$f = \alpha_0(\varepsilon) f_0 + \alpha_1(\varepsilon) f_1 + \ldots$

- Substitute in & solve for each $f_0, f_1, f_2, \ldots$

e.g. $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$

If its a regular problem. Do the above & it will work. Problem for singular situations.

If you set $\varepsilon = 0$, lose leading order terms.

$\Rightarrow$ so RESCALE to keep those important terms

Choice of rescaling

- Balance terms in equations or integrals (Sheet 1 Q4)
- Solve for $\varepsilon = 0$ & look for where terms are larger or smaller than expected (Sheet 4 Q3 & Q4)
- Solve leading & 1st order & see where the correction term becomes large.

E.g. $f \sim 1 + \varepsilon x$

$x = \Theta(1) \to$ ✓

$x = \Theta(\frac{1}{\varepsilon}) \to$ correction term too big.

General Asymptotics $\Rightarrow$ specialise

Integrals w/ exponentials

$$I(\lambda) = \int_A^B f(x) e^{\lambda g(x)} dx \quad \text{as } \lambda \to \infty$$

w/ integral, more information than you need. Different integrands give the same number (the integral - more degrees of freedom).

Look at where integrand exponentially dom

Endpoints:

Watson's Lemma

$I(\lambda) \sim \frac{1}{\lambda} f(A) e^{-\lambda g(A)}$

(simple cases)

interior:

Laplace's method: $g'(x_0) = 0$

$I(\lambda) \sim \sqrt{\frac{2\pi}{-\lambda g''(x_0)}} f(x_0) e^{\lambda g(x_0)}$

Steepest descent:

deform contour s.t. $g'(z_*) = 0$ (saddle pt)

Then use laplace's method

Stationary phase

$I(\lambda) = \int_A^B f(x) e^{i\lambda \tilde{g}(x)} dx$ highly oscillatory

$\Rightarrow$ set $i\tilde{g}(x) = g(x)$ & use steepest descent

Integral Transforms

extend $f:[0,\infty) \to \mathbb{C}$ to $g: \mathbb{R} \to \mathbb{C}$ by zero extending
$$g(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transforms

'forward' $$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

inverse $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t}\, d\omega$$

Fourier Properties

name	$g(t)$	$\tilde{g}(\omega)$
Shift 1	$f(t-a)$	$e^{-i\omega a} \tilde{f}(\omega)$
Shift 2	$e^{i\Omega t} f(t)$	$\tilde{f}(\omega - \Omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} \tilde{f}(\frac{\omega}{a})$
Diff	$f'(t)$	$i\omega \tilde{f}(\omega)$
$\times t$	$t f(t)$	$i \frac{d\tilde{f}}{d\omega}(\omega)$
convolution	$\int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$	$\tilde{f}(\omega) \tilde{g}(\omega)$

Plancherel's Theorem

$$\underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert \tilde{f}(\omega) \rvert^2 d\omega}_{\text{energy in frequency density}} = \underbrace{\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt}_{\text{energy in time signal}}$$

Laplace Transforms

$$\hat{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty + a}^{i\infty + a} \hat{f}(s) e^{st}\, ds$$

a past all singularities

Laplace Properties

name	$g(t)$	$\hat{g}(s)$
Shift 1	$f(t-a)$	$e^{-sa} \hat{f}(s)$
Shift 2	$e^{at} f(t)$	$\hat{f}(s-a)$
Scaling	$f(at)$	$\frac{1}{a} \hat{f}(\frac{s}{a})$
Diff	$f'(t)$	$-f(0) + s\hat{f}(s)$
	$f''(t)$	
$\times t$	$t f(t)$	$-\frac{d\hat{f}}{ds}(s)$
	t^n	$\frac{n!}{s^{n+1}}$
convolution	$\int_0^{\infty \leftarrow \text{or } t} f(\tau) g(t-\tau)\, d\tau$	$\hat{f}(s) \hat{g}(s)$

Examples

$f(t)$	$\hat{f}(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos(at)$	$\frac{s}{s^2 + a^2}$
$\sin(at)$	$\frac{a}{s^2 + a^2}$
t^n	$\frac{n!}{s^{n+1}}$

$$\frac{d\hat{y}}{dt} = -y(0) + s\hat{y}(s)$$ JUST DIFF

$$\frac{d^2\hat{y}}{dt^2} = -\dot{y}(0) + s\frac{d\hat{y}}{dt}$$
$$= -\dot{y}(0) + s(-y(0) + s\hat{y}(s))$$
$$= -\dot{y}(0) - sy(0) + s^2\hat{y}(s)$$