GEOMETRY

Euclidian Geometry

Euclidian metric

$d(x,y) = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{\frac{1}{2}} = \|x-y\|$

$\langle x,y\rangle = \sum x_i y_i$

$\|x\| = \sqrt{\langle x,x\rangle}$

Cauchy Schwartz

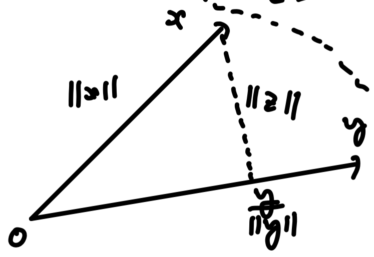
$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$

Pf: Assume $y\neq 0$

WTS: $\frac{|\langle x,y\rangle|}{\|y\|} \le \|x\|$

$\Leftrightarrow |\langle x, \frac{y}{\|y\|}\rangle| \le \|x\|$

$z = x - \frac{\langle x,y\rangle}{\|y\|^2}y$



then $0 \le \|z\|^2 = \langle z,z\rangle = \cdots = \|x\|^2 - \left(\frac{\langle x,y\rangle}{\|y\|}\right)^2$

Bilinearity of Inner Product

$\langle a+b, c\rangle = \langle a,c\rangle + \langle b,c\rangle$

$\langle \lambda a, b\rangle = \lambda\langle a,b\rangle$

$\langle a,b\rangle = \langle b,a\rangle$

Pf: duh!

Metric Definitions

Non-degenerate, Symmetric, triangle inequality

(i) $d(x,y)=0 \Leftrightarrow x=y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \le d(x,z) + d(z,y)$

Translation Invariance

$d(x,y) = d(x-z, y-z)$

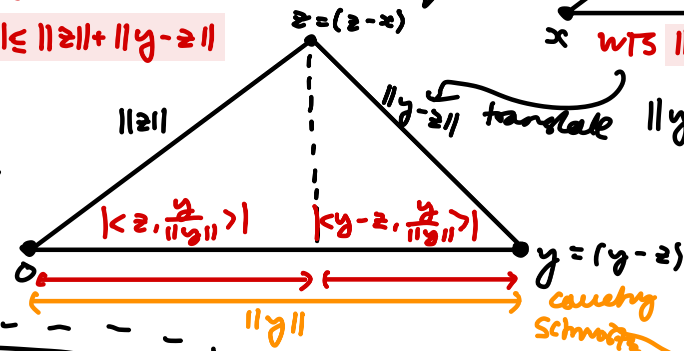
Pf: $d(x,y) = \|x-y\|$
$= \|(x-z)-(y-z)\|$
$= d(x-z, y-z)$

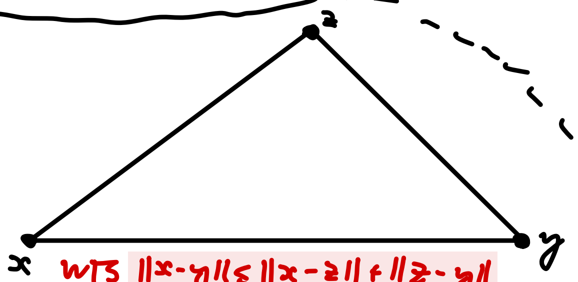
Euclidian metric is a metric

$d(x,y) = \|x-y\|$ is a metric on $\mathbb{R}^n$

Pf: Non-degen...
Symmetric...
Triangle inequality

WTS: $\|y\| \le \|z\| + \|y-z\|$





WTS $\|x-y\| \le \|x-z\| + \|z-y\|$

$\|y\| = \frac{\langle y,y\rangle}{\|y\|} = \frac{1}{\|y\|}\langle z+y-z, y\rangle$

$= \frac{1}{\|y\|}\left[\langle z,y\rangle + \langle y-z,y\rangle\right]$

$= \langle z, \frac{y}{\|y\|}\rangle + \langle y-z, \frac{y}{\|y\|}\rangle$

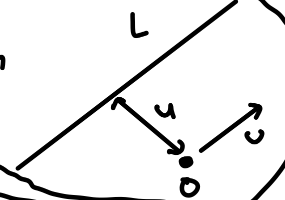
$\le |\langle z, \frac{y}{\|y\|}\rangle| + |\langle y-z, \frac{y}{\|y\|}\rangle|$

$\le \|z\|\cdot\left\|\frac{y}{\|y\|}\right\| + \|y-z\|\cdot\left\|\frac{y}{\|y\|}\right\|$

$= \|z\| + \|y-z\|$

Line

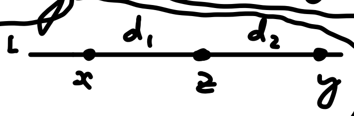
$L = \{u+\lambda v \mid \lambda\in\mathbb{R}\} \subset \mathbb{R}^n$



Formula collinearity

$z = x + \lambda(y-x)$
$= (1-\lambda)x + \lambda y$
$= \lambda_1 x + \lambda_2 y$

$\lambda_1 + \lambda_2 = 1, \quad |\lambda_1| : |\lambda_2| = d_1 : d_2$



Pf: $x = u + \lambda_x v$
$y = u + \lambda_y v$
$z = u + \lambda_z v$

$\Rightarrow z = k_1 x + k_2 y$ — solve for u & v

Distance Preserving map

(X_1, d_1), (X_2, d_2) metric spaces

$f: X_1 \to X_2 \quad \forall P,Q\in X_1$

$d_2(f(P), f(Q)) = d_1(P,Q)$

Isometry

bijective distance preserving map

Motion

Isometry of Euclidian Space

Classifying Isometries of $\mathbb{E}^n$.

STEP1 collinearity in terms of the metric

$|\langle x,y\rangle| = \|x\|\cdot\|y\| \Leftrightarrow x = \lambda y$

Pf: equality $\Leftrightarrow z = 0 = x - \frac{\langle x,y\rangle}{\|y\|^2}y$

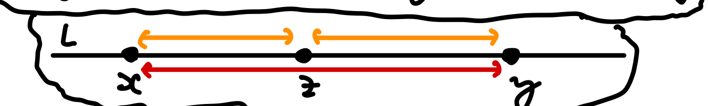
collinearity preserved by translations

x,y,z colinear $\Leftrightarrow z = (1-\lambda)x + \lambda y$
$\Leftrightarrow 0 = (1-\lambda)(x-z) + \lambda(y-z)$
$\Leftrightarrow x-z, y-z, 0$ colinear

collinearity in terms of the metric

(= in Δ inequality)

x,y,z colinear $\Leftrightarrow d(x,y) = d(x,z) + d(z,y)$



Pf: '$\Rightarrow$' $z = x + \lambda(y-x)$, $\lambda\in(0,1)$ 'z in the middle'

$d(x,z) + d(z,y) = \|x-z\| + \|z-y\|$
$=$ [sub in z]
$= \lambda\cdot\|y-x\| + (1-\lambda)\|y-x\|$
$= \|y-x\|$

'$\Leftarrow$'

$d(x,y) = d(x,z) + d(z,y)$

$d(x',y') = d(x',0) + d(0,y')$ [translate & switch]

$\Rightarrow \|x-y\| = \|x\| + \|y\|$

$\Rightarrow \|x-y\|^2 = \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$

$\langle x-y, x-y\rangle =$ "

$\hookrightarrow = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$

$\Rightarrow -2\langle x,y\rangle = 2\|x\|\cdot\|y\|$

$\Rightarrow |\langle x,y\rangle| = \|x\|\cdot\|y\|$ ⟵ equality in cauchy schwartz!

$\Rightarrow y = \lambda x$

$\Rightarrow y = \lambda x + (1-\lambda)0$

$\Rightarrow x, y, 0$ colinear

$\Rightarrow x, y, z$ colinear after translation & relabling.

STEP2 Isometries map lines to lines

x,y,z colinear $\Rightarrow T(x), T(y), T(z)$ colinear

$z = (1-\lambda)x + \lambda y \Rightarrow T(z) = (1-\lambda)T(x) + \lambda T(y)$

Pf: x,y,z colinear
$\Updownarrow$ can permute
$d(x,y) = d(x,z) + d(z,y)$

$\Rightarrow d(T(x), T(y)) = d(T(x), T(z)) + d(T(z), T(y))$

$\Rightarrow T(x), T(y), T(z)$ colinear

linear map: $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$

Affine map: $T(x) = L(x) + b$

STEP3 isometries as linear maps

① T is affine
② $x \longmapsto T(x) - b$ is linear
T linear if $T(b) = 0$
③ $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$

Every euclidian isometry $T: \mathbb{R}^m \to \mathbb{R}^n$ is affine. $n = m$ [$T = L + b$]

STEPS Linear isometries have orthogonal matricies

$L: \mathbb{R}^n \to \mathbb{R}^n$ linear map, matrix A wrt E.
① L is an isometry
② $\|L(x)\| = \|x\|$ [L preserves norm]
③ $\langle L(x), L(y)\rangle = \langle x,y\rangle$ [L preserves $\langle\cdot,\cdot\rangle$]
④ Matrix A is orthogonal. $AA^T = I_n$

Pf: (3) $\Rightarrow$ (4)

$(AA^T)_{ij} = \sum_{k=1}^{n}(A^T)_{ik}A_{kj}$

$= \sum_{k=1}^{n} A_{ki}A_{kj}$

$= \sum_{k=1}^{n} L(e_i)_k L(e_j)_k$

$= \langle L(e_i), L(e_j)\rangle$

e_i is standard basis $\Rightarrow \langle L(e_i), L(e_j)\rangle$

$= \langle e_i, e_j\rangle$

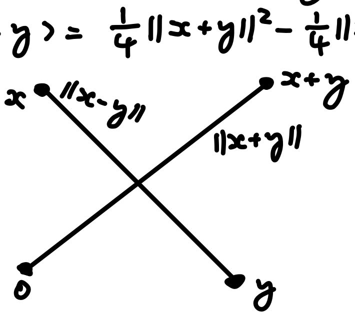
$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}$

$\Rightarrow A$ orthogonal.

Euclidian isometry [motion] $T: \mathbb{R}^n \to \mathbb{R}^n$ has form $T(x) = Ax + b$, A orthogonal

Polarisation identity

$\langle x,y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$



Normal Form Theorem

$L: \mathbb{R}^n \to \mathbb{R}^n$, $AA^T = I_n$, $L(x) = Ax$.
$\exists$ orthonormal basis of $\mathbb{R}^n$ s.t. L has matrix

$$\begin{bmatrix} I_k & & & & \\ & -I_m & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_l \end{bmatrix} \qquad B_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

Pf: Extend to complex #s long!

Isometries of $\mathbb{R}^2$

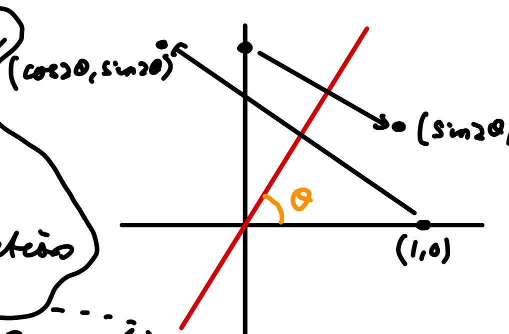
matrix of rotation anticlockwise through θ

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

matrix of reflection in line at θ to x axis

$$\rho_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$T: \mathbb{E}^2 \to \mathbb{E}^2$ isometry, $T(x) = Ax + b$ where wrt $\{e_1, e_2\}$, $A = R_\theta$ or ρ_θ, $b \in \mathbb{R}^2$.
[T is a rotation/reflection then translation].



Pf: $T(x) = Ax + b$, A orthogonal.
$A = [v, w]$. $AA^T = I_2$
$\Rightarrow \|v\| = \|w\| = 1$
$\langle v, w\rangle = 0$
let $v = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a^2 + b^2 = 1$
$\Rightarrow (a,b) = (\cos\theta, \sin\theta)$
$\Rightarrow w = \pm(-b, a)$
$\Downarrow$
two cases...

The Group of Euclidean isometries

The orthonormal group $O(n, \mathbb{R})$

$O(n) = \{A \in \mathbb{R}^{n \times n} : AA^T = I_n\}$

Pf: closure $A, B \in O(n)$
$(AB)(AB)^T = ABB^TA^T$
$= A I_n A^T$
$= I_n$

associative true by matrix associativity

identity $I_n I_n^T = I_n$ and

inverse $AA^T = I_n \Rightarrow A^T = A^{-1}$
$\det(AA^T) = 1$
$\det(A)\det(A^T) = 1$
$\Rightarrow \det(A) \neq 0$
so inverse $\forall A \in O(n)$

Group of isometries of a metric space $\mathrm{Isom}(X, d)$

(X, d) a metric space
$\mathrm{Isom}(X,d) = \{ f : X \to X \mid f \text{ is an isometry of } X\}$

$\mathrm{Isom}(X,d)$ is a group

Pf: Isometries have the form $L(x) = Ax + b$

closure $A_2(A_1 x + b_1) + b_2$
$\underbrace{A_2 A_1}_{\text{orthogonal}} x + \underbrace{A_2 b_1 + b_2}_{\text{vector}}$
$\underbrace{\qquad\qquad\qquad}_{\text{isometry}}$

identity I_n is an isometry. $b = 0$

inverse WTS inverse of an isometry is an isometry. $L(x) = Ax + b$
$\Rightarrow L^{-1}(x) = A^T(x - b)$
$= A^T x - A^T b$

associativity $A \circ (B \circ C) = (A \circ B) \circ C$
composition of isometries associative

$$\boxed{\mathrm{Isom}(\mathbb{R}^n, d) \cong O(n, \mathbb{R}) \times \mathbb{R}^n}$$

need a semi-direct product [non-examb]

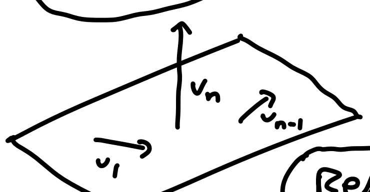
Reflections Generate $\mathrm{Isom}(\mathbb{E}^n)$

Hyperplane of $\mathbb{R}^n$

Affine subspace of $\mathbb{R}^n$ of dimension $n-1$. Has form

$\Pi = V + b = \{b + v : v \in V\}$
$= \{ b + \sum_{i=1}^{n-1} \lambda_i v_i \mid \lambda_i \in \mathbb{R}\}$
$= (\mathbb{R} v_n)^\perp + b$
$= \{ v \in \mathbb{R}^n : \langle v, v_n \rangle = \beta\}$

$b \in \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$ is a $n-1$ dim subspace with basis $v_1, \dots, v_{n-1}$.
v_n orthogonal to V
$\beta = \langle v_n, b\rangle$

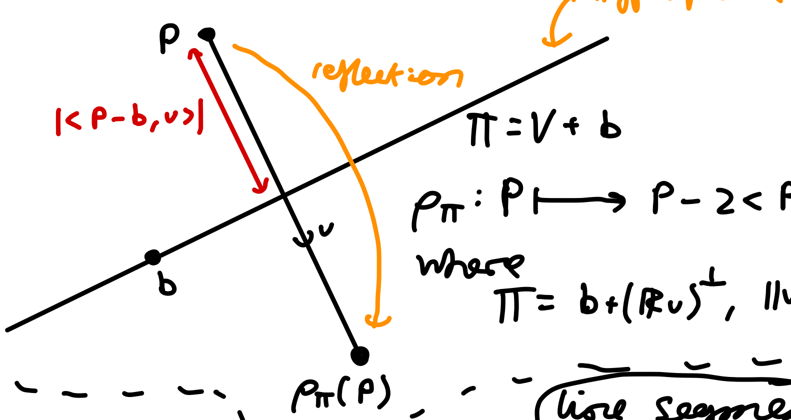


Fixed points of a map T

$\mathrm{Fix}(T) = \{ x \in \mathbb{R}^n : T(x) = x\}$

Reflection in Π

is an Euclidean isometry $\rho_\Pi : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\mathrm{Fix}(\rho_\Pi) = \Pi$



$\Pi = V + b$

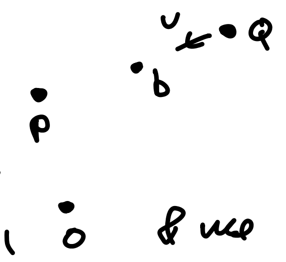
$\rho_\Pi : P \longmapsto P - 2\langle P - b, v\rangle v$

where $\Pi = b + (\mathbb{R}v)^\perp$, $\|v\| = 1$

Pf: $\rho_\Pi(\rho_\Pi(P)) = P$

Given distinct points $P, Q \in \mathbb{R}^n$ $\exists$ reflection ρ s.t. $\rho(P) = Q$

Pf:



let $v = \dfrac{P - Q}{\|P - Q\|}$

let $b = \frac{1}{2}P + \frac{1}{2}Q$

$\Pi = (\mathbb{R}v)^\perp + b$

& use $P \longmapsto P - 2\langle P - b, v\rangle v$

The group $\mathrm{Isom}(\mathbb{E}^n)$ is generated by reflections

f Any isometry of $\mathbb{E}^n$ is the product of at most $n-1$ reflections.

Pf: decompose the normal form into a composition of reflections.

$$J_i = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{etc...}$$

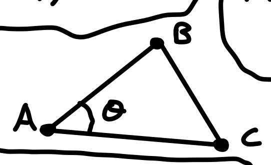
Examples of decompositions into reflections in notes, p. 28 of pdf

Line segment

$A, B \in \mathbb{R}^n$. line seg
$[A, B] := \{(1-\lambda)A + \lambda B : \lambda \in [0,1]\}$

Triangle

$A, B, C \in \mathbb{R}^n$
$\Delta(ABC) = [AB] \cup [BC] \cup [CA]$



$a \cdot b = |a||b|\cos\theta$
$\Rightarrow \cos\theta = \dfrac{a \cdot b}{|a||b|}$

Angles in a Δ sum to π

Angle

measure of angle $\angle(BAC)$ to be the $\theta \in [0, \pi]$ s.t.

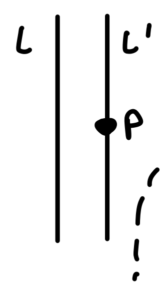
$$\cos\theta = \frac{\langle B - A, C - A\rangle}{\|B - A\|\|C - A\|} \in [-1, 1]$$

Angles preserved by isometry

Pf: defined in terms of inner product.

Euclid's 5th postulate

(Can prove instead of taking as an axiom)



Given line L, point $P \in \mathbb{R}^2$, $P \notin L$,
$\Rightarrow$ unique line L' with $P \in L'$, $L \cap L' = \emptyset$

Pf: Given $L = \mathbb{R}v + b$, let $L' = \mathbb{R}v + P$ $P \in L'$
Suppose $w \in L \cap L'$.
$\Rightarrow w = \alpha v + b = \beta v + P \Rightarrow P = (\alpha - \beta)v + b \in L$
so $P \in L$ $\times$ $\Rightarrow L \cap L' = \emptyset \Rightarrow L'$ exists.

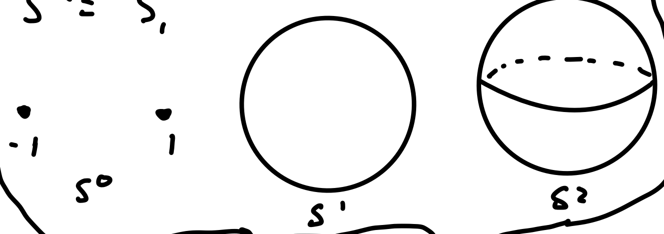
uniqueness Suppose another $L'' = \mathbb{R}v' + P$. $P \in L''$
with $L \cap L'' = \emptyset$. $w \in L \cap L''$
$\Rightarrow w = \alpha v + b = \beta v' + P$
$\Rightarrow b - P = \beta v' - \alpha v = (v \ v')\begin{pmatrix} -\alpha \\ \beta \end{pmatrix}$
require $(v \ v')$ matrix invertible.
$\Leftrightarrow$ v and v' linearly independent.
$\Rightarrow$ If $L \cap L'' = \emptyset$
$\Rightarrow v = \lambda v'$?

Spherical Geometry

n-dimensional sphere

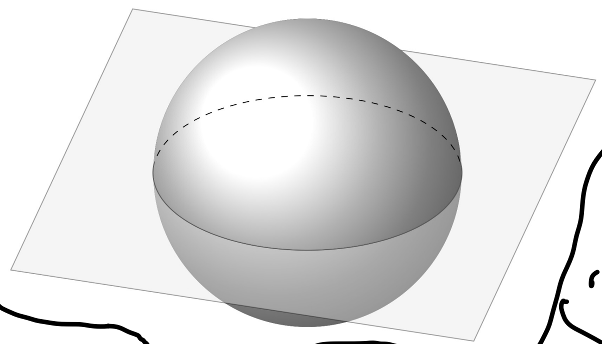
$S_r^n = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$

$S^n = S_1^n$



great circle

A spherical line/great circle is a intersection of S_r^n w/ a 2-dim vector subspace of $\mathbb{R}^{n+1}$



antipodal points

$P, Q \in S_r^n$ antipodal if $Q = -P$

Unique great circles

If $P, Q \in S_r^n$ are not antipodal, then theres a unique great circle containing both of them.

Pf: $\|P\| = \|Q\| = r \neq 0$. Not antipodal $\Rightarrow$ not multiples of each other

$\mathbb{R}v$ intersects S_r^n at pts s.t. $\|\lambda v\| = |\lambda|\|v\| = R \Rightarrow \lambda = \pm \frac{R}{\|v\|}$ only two solutions

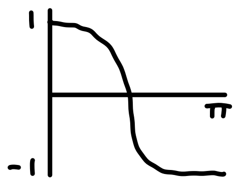
$\Rightarrow \mathbb{R}P \oplus \mathbb{R}Q = \{\alpha P + \beta Q : \alpha, \beta \in \mathbb{R}\}$ is the only two dim subspace containing both of them.

Spherical distance

between $P, Q \in S_r^n$ is length of the shortest arc of a great circle joining them

Spherical metric

$$d_{S_r^n}(P,Q) = r\cos^{-1}\left(\frac{\langle P, Q\rangle}{r^2}\right)$$



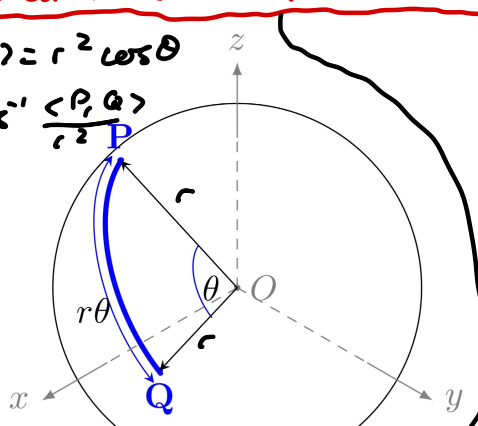
Take $\cos^{-1} \in [0, \pi]$

Will take $r = 1$ for simplicity

$\langle P, Q\rangle = r^2 \cos\theta$

$\Rightarrow \theta = \cos^{-1} \frac{\langle P, Q\rangle}{r^2}$

"$r\theta = r$"



Pf: WLOG, can assume

$P = (0,0,1)$

$Q = (\sin\beta, 0, \cos\beta)$

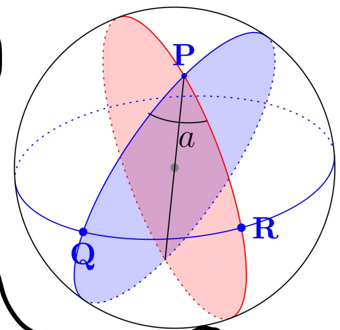
$R = (\sin\gamma\cos a, \sin\gamma\sin a, \cos\gamma)$

$\cos\alpha = \langle Q, R\rangle$

$= \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a$

Goal: Show $\mathrm{Isom}(S^n, d) \cong O(n+1)$

Spherical triangle



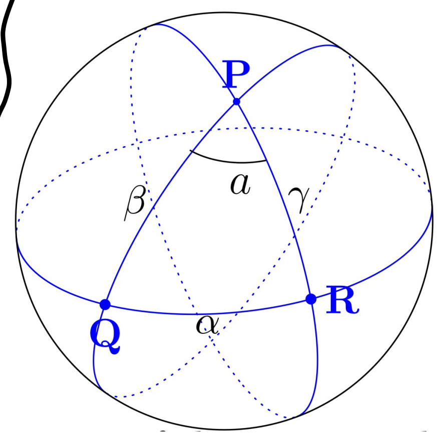
3 distinct S^n vertices P, Q, R and 3 great arcs.

8 triangles w/ these vertices must specify!

Spherical angle

at P between lines PQ and PR is the dihedral (angle between planes) angle between containing these lines (or angle between tangent lines to the sphere that lie in these to planes & touch the sphere at P.

Main formula of Spherical trionometry



$\alpha = d(Q, R)$

$\beta = d(P, Q)$

$\gamma = d(P, R)$

a is spherical angle between PQ and PR. Then

$$\cos\alpha = \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a$$

Spherical distance Satisfies triangle inequality

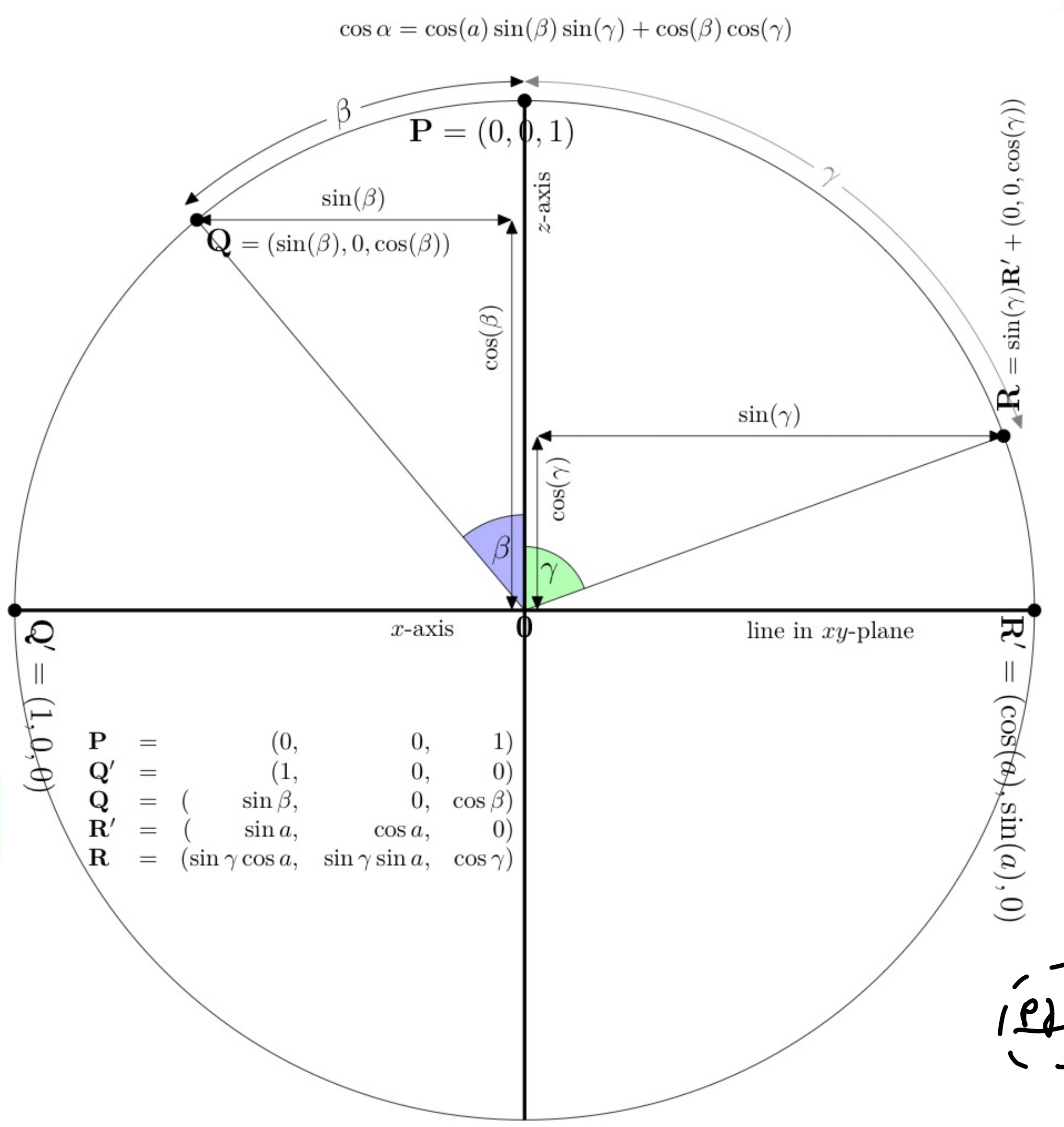
$\alpha \leq \beta + \gamma$ w/ equality

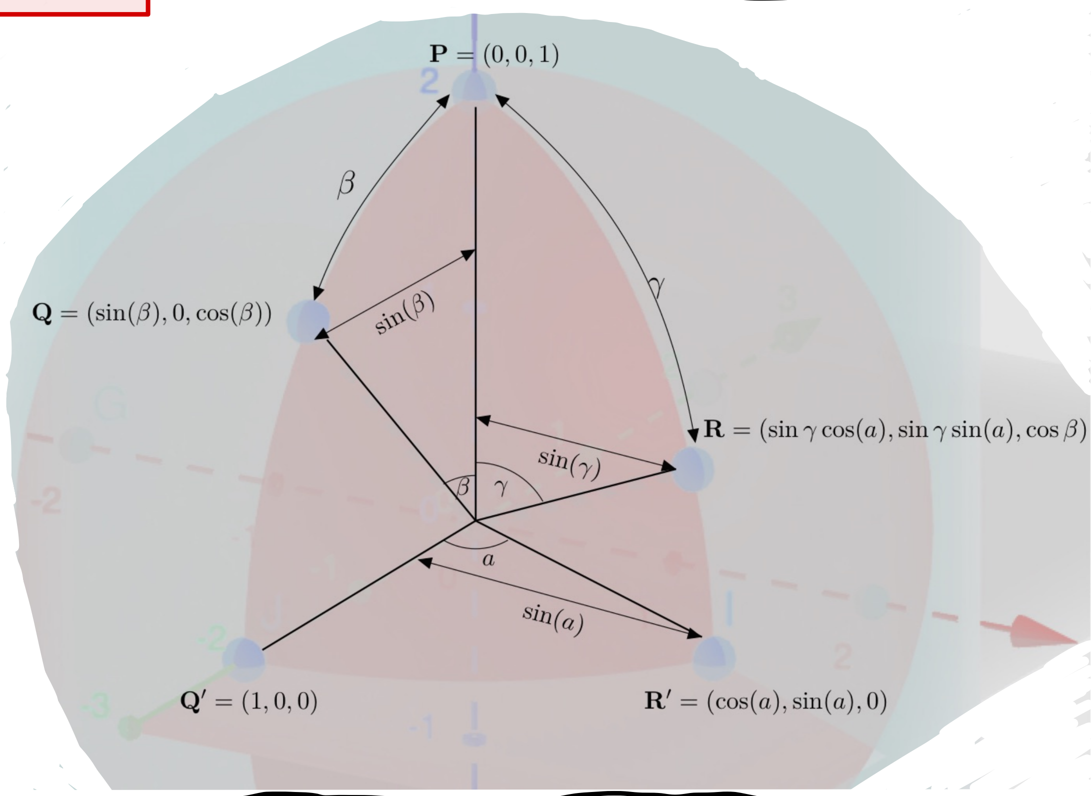
$\alpha + \beta + \gamma = 2\pi \iff \alpha = \pi$

& $a = 0 \iff \alpha = |\beta - \gamma|$

$\Rightarrow$ metric

Pf: use dbl angle formula





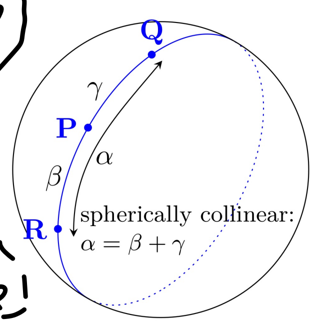
condition for points lying on a great circle

$P, Q, R \in S^n$

$\alpha = \beta \pm \gamma$ or $\alpha + \beta + \gamma = 2\pi$ $\iff$ P, Q, R lie on a great circle

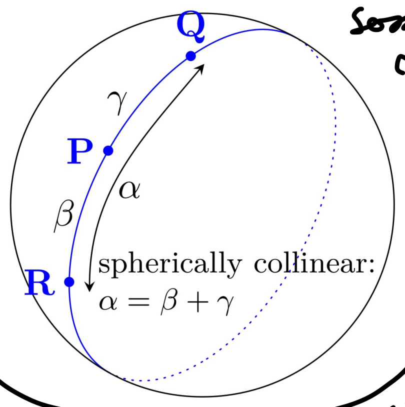
Pf: $\alpha + \beta + \gamma = 2\pi \iff a = 0$ or π

$\iff P, Q, R$ lie on a great circle!



Spherically collinear

P, Q, R collinear on $S^n \iff$ if all contained in some great circle C



Isometries preserve antipodal points

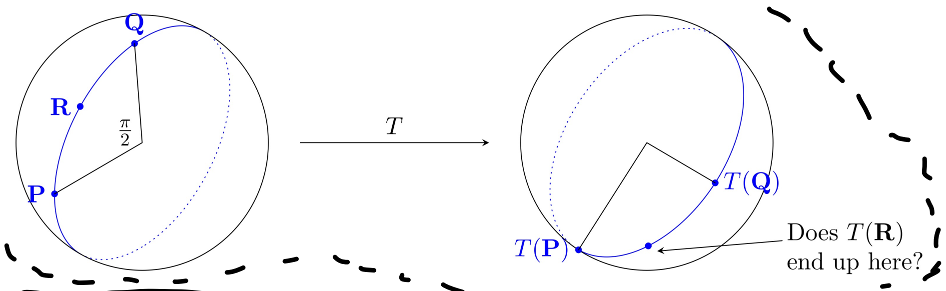
$T: S^n \longrightarrow S^n$ isometry preserves antipodal pts

Pf: P, Q antipodal $\iff Q = -P$
$\iff \langle P, Q \rangle = -1$
$\iff d(P, Q) = \pi$
$\iff d(T(P), T(Q)) = \pi$
$\iff T(P), T(Q)$ antipodal.

$|\langle P, Q \rangle| = \|P\|\|Q\|$
$\iff P \in \mathbb{R} Q$
$\|P\| = \|Q\|$

An Isometry $T: S^n \longrightarrow S^n$ preserves great circles

Pf:



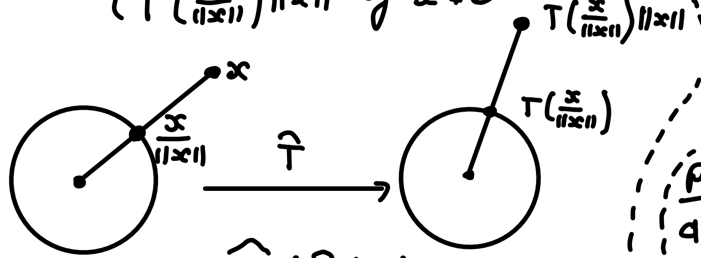
A bijective map $f: \mathbb{R}^k \longrightarrow \mathbb{R}^k$ which preserves the standard inner product on $\mathbb{R}^k$ is a linear isometry of $\mathbb{R}^k$

Pf:
$d(x,y) = \sqrt{\langle x-y, x-y \rangle}$
$\|f(y) - f(x)\|^2 = \cdots \|x - y\|$

If T is an isometry of S^n $\Downarrow$ T extends to an Euclidian isometry $\hat{T}$ of $\mathbb{R}^{n+1}$

[$\exists\, \hat{T}$ of $\mathbb{R}^{n+1}$ euclidian isometry & $\hat{T}|_{S^n} = T$]

Pf: Given T isom of S^n,
def
$$\hat{T} = \begin{cases} 0 & \text{if } x = 0 \\ T\left(\frac{x}{\|x\|}\right)\|x\| & \text{if } x \neq 0 \end{cases}$$



bijection: $\hat{T}^{-1}(\hat{T}(x)) = x$

preserves inner product: $\langle \hat{T}(x), \hat{T}(y) \rangle = \langle x, y \rangle$

$\Rightarrow$ Euclidian isometry
[distance in terms of $\langle \cdot, \cdot \rangle$]

Every isometry $T: S^n \to S^n$ is of the form $T(x) = Ax \ \forall x \in S^n$ for some orthogonal $A \in O(n+1)$

Pf: T is a restriction of an Euclidian isometry to $S^n \Rightarrow$ result.

If T Euclidian isom of $\mathbb{R}^{n+1}$, then $T|_{S^n}$ is a spherical isom