week 1 - Intro to dynamical Systems & stability

A system of ODEs

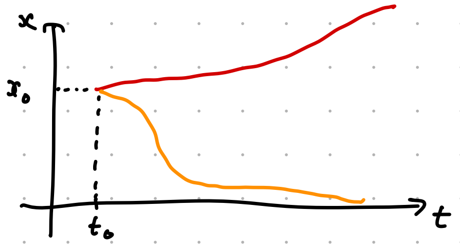
$$\dot{x}_1 = f_1(x_1,\dots,x_n,t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1,\dots,x_n,t)$$

$\frac{dx_1}{dt}$

Autonomous systems so our $f_1,\dots,f_n$ doesn't depend on t and $n \le 3$ simplification!

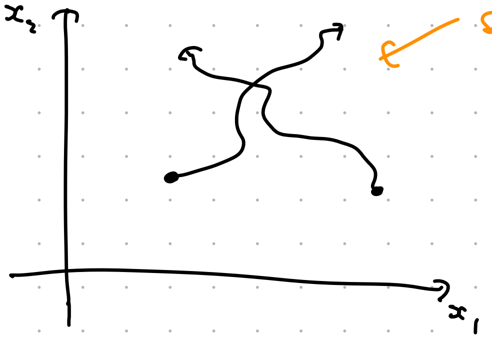
Assume existence and uniqueness of solutions, this means:

- 1D:

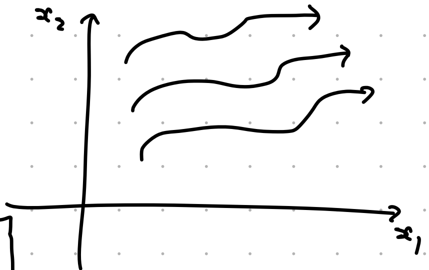


either increasing or decreasing but not both or contradictions.

- 2D:



solutions can't cross in 2d. (contradicts uniqueness) we get that trajectories never cross

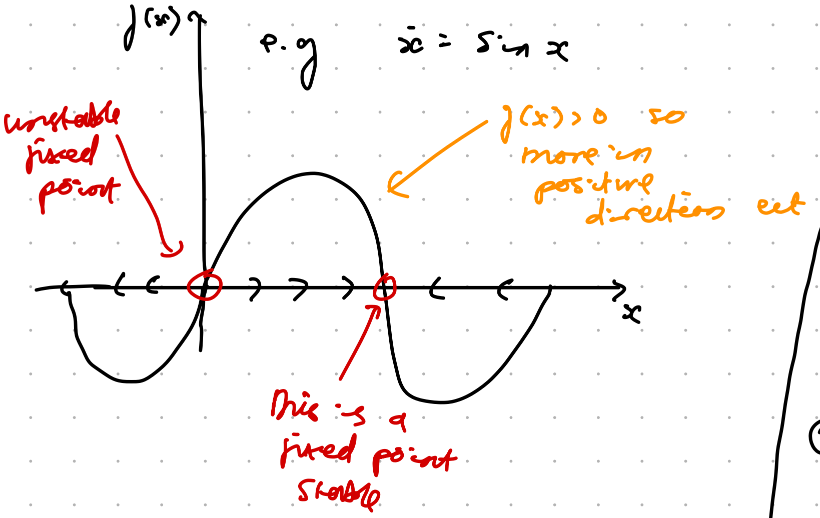


Def: $x_0 \in \mathbb{R}^n$ is an initial condition & the path given by $x(t) \in \mathbb{R}^n$ is called the trajectory corresponding to x_0.

First order Systems

$\dot{x} = f(x)$ ← independent variable doesn't appear ∴ autonomous.

e.g. $\dot{x} = \sin x$



"phase line"/"vector field" = phase diagram.

Def: A fixed point x^* is stable if all sufficiently small deviations from x^* damp out over time.

① If all solutions that start near x^* stay near x^* forever, we say that x^* is Lyapunov stable. simplest type →

② If all solutions that start near x^* converge to x^* (as $t \to \infty$) then x^* is asymptotically stable. no mention of how long

③ If solutions converge to x^* at some minimum rate, then

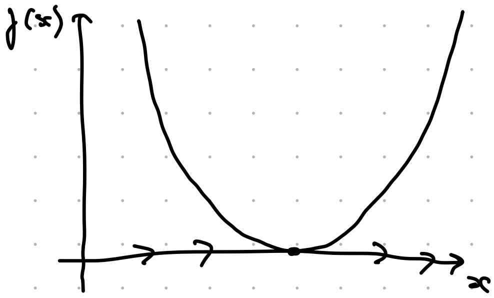
Def: x^* is unstable if disturbances grow in time.

x^* is exponentially stable.

other options!

Saddle node occurs when $\boxed{\dot{x} = f(x)}$

more complex to classify.



Stable from one side & unstable from the other.

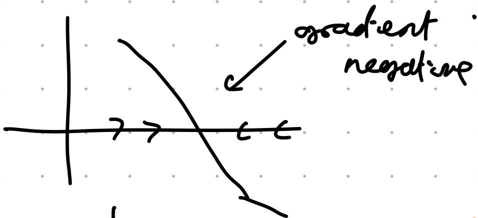
Linear stability Analysis

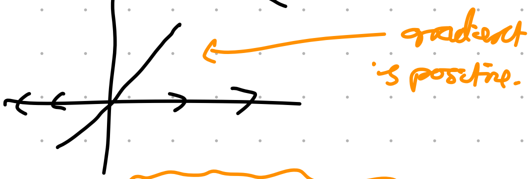
Look at a system, stable or not???

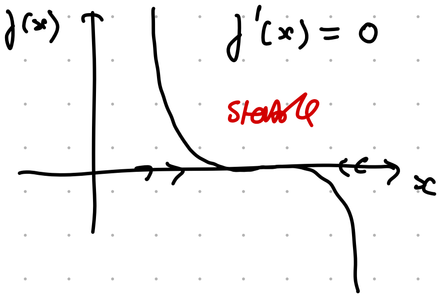
Thm: Take $n=1$

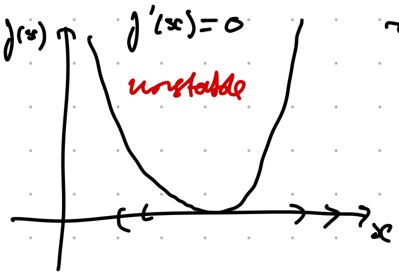
① x^* is stable if $\frac{df}{dx}(x) < 0$

② x^* is unstable if $\frac{df}{dx}(x) > 0$









Just draw a picture if you have $f'(x)=0$

Example: (Population growth)

- N is population size, $(N > 0)$
- $r > 0$, $k > 0$
-

$$\dot{N} = rN\left(1 - \frac{N}{k}\right) = f(N)$$

k is carrying capacity
- If $k > N$, then decrease
- If $k < N$, then increase

$\boxed{f(N) = 0}$ ← when?

when $N = 0$ or $N = k$

[This is the logistic equation]

Now, want $f'(N) = r - 2r\frac{N}{k}$

$f'(0) = r > 0 \Rightarrow$ unstable [no one in population so unstable & will grow]

$f'(k) = -r < 0 \Rightarrow$ stable [start near carrying capacity, then will stay]

Bifurcations

change value of a parameter but something qualitatively changes.

- change in # of fixed points.
- change/switch in stability of fixed points.

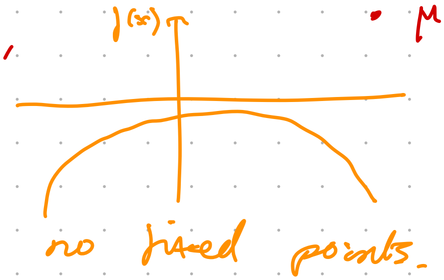
There are different types.

① Saddle node bifurcation:

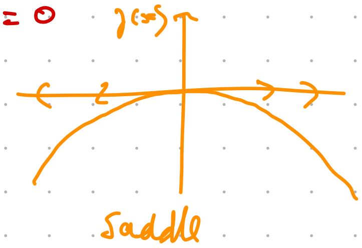
0 fixed points $\longrightarrow$ 1 fixed point $\longrightarrow$ 2 fixed points one stable & one unstable.

Example: $\dot{x} = \mu - x^2$. Consider different μ

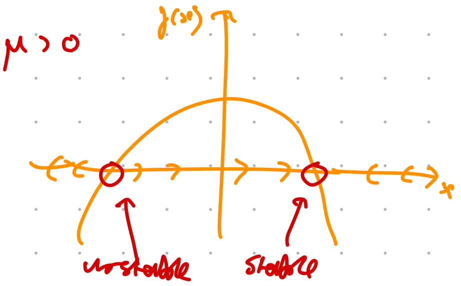
- $\mu < 0$,



- $\mu = 0$



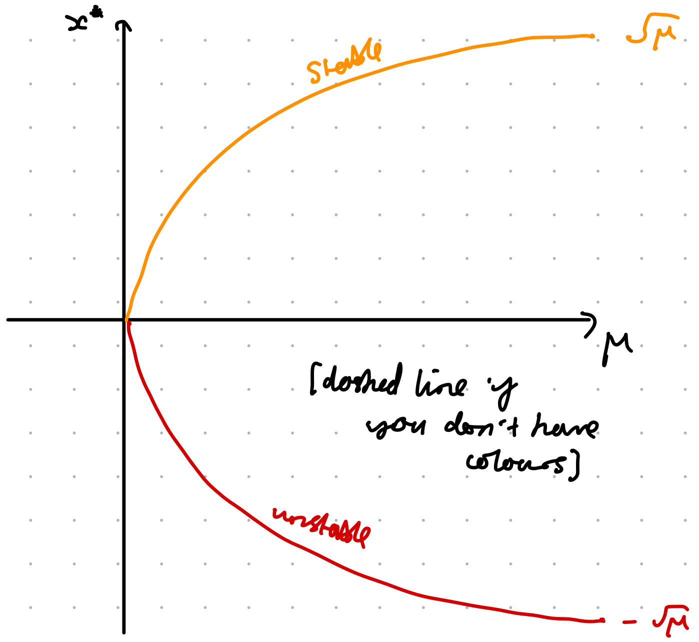
- $\mu > 0$



Above is messy, instead, draw a bifurcation diagram, can vary along horizontal axis, then value of fixed points on y

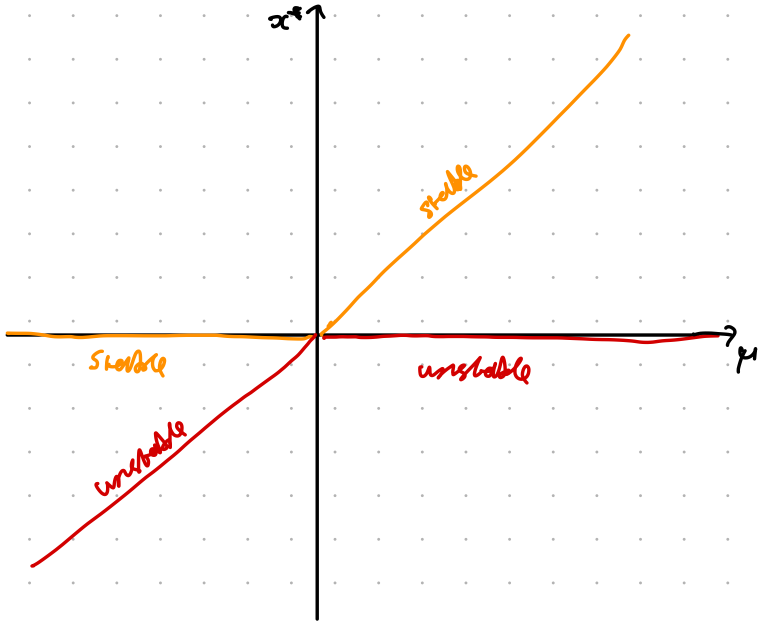
$\dot{x} = \mu - x^2 = 0$

$\sqrt{\mu} = x$



[dashed line if you don't have colours]

saddle node bifurcation.



Transcritical bifurcation.

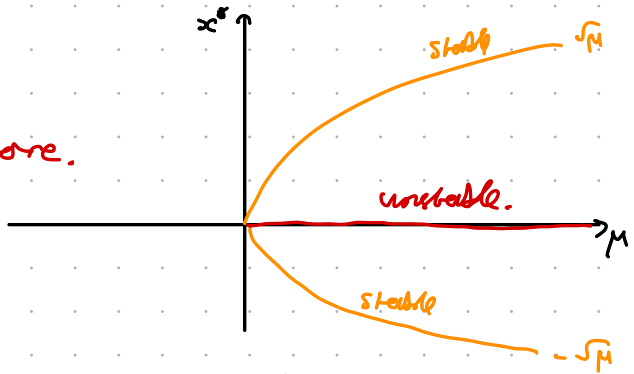
② Transcritical bifurcation

1x stable
1x unstable $\longrightarrow$ 1x saddle $\longrightarrow$ 1x unstable
1x stable

Example: $\dot{x} = \mu x - x^2 = 0$

$x^* = 0, \mu$ $\Rightarrow$ see picture above.

③ Pitchfork bifurcation (looks like one!)



Example: $\dot{x} = \mu x - x^3 = x(\mu - x^2)$

$x^* = 0,\ x^* = \pm\sqrt{\mu}$ so depending on μ, get different # fixed points.
if $\mu > 0$, if $\mu = 0$ one... stable for $\mu > 0$

← number of equations

Second Order Systems

$$\left\{ \begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{array} \right\}$$

System now because 2.

where will we start from initial point?

$$\underline{v}(x,y) = (\dot{x}, \dot{y}) = (f(x,y), g(x,y))$$

Linear Systems

zero is always a fixed point in a linear system.

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$\boxed{\dot{\underline{x}} = A\underline{x}}$ $\underline{x} = (x,y)$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\underline{x} \in \mathbb{R}^2$ here

$a, b, c, d \in \mathbb{R}$

Example: $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \dot{\underline{x}} = A\underline{x} \Rightarrow \boxed{\dot{x} = ax \quad \dot{y} = -y}$

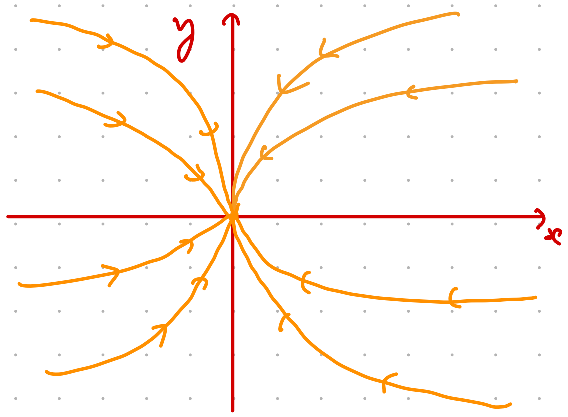
as simple as a system can be

$x(t) = x_0 e^{at}$ $y(t) = y_0 e^{-t}$

Now cases:

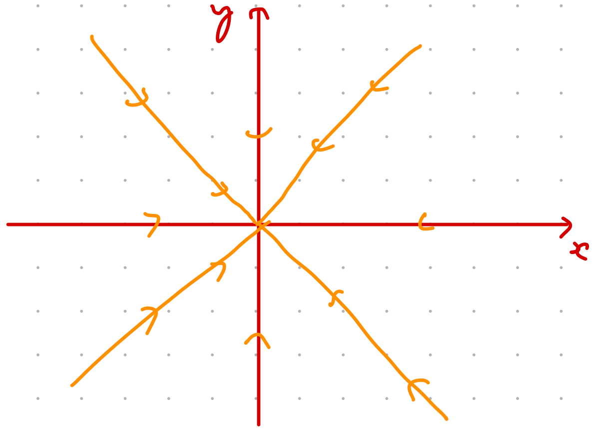
① $\underline{a < 0}$

(i) $a < -1$ $x(t) \longrightarrow 0$ quicker than $y(t) \longrightarrow 0$

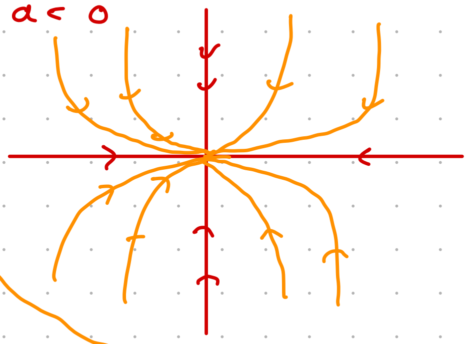


stable node

(ii) $a = -1$ stable star



(iii) $-1 < a < 0$



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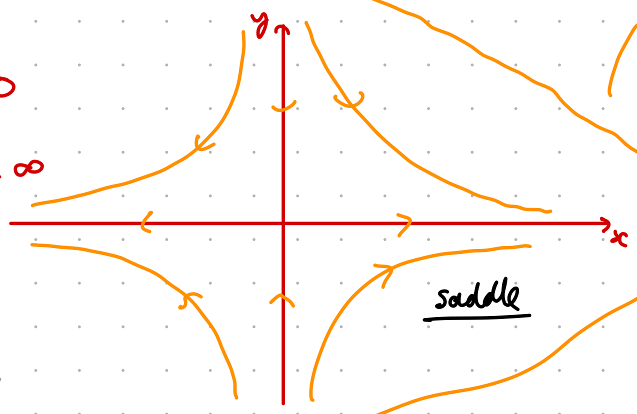
② $\underline{a > 0}$

$y(t) \longrightarrow 0$

$x(t) \longrightarrow \pm\infty$



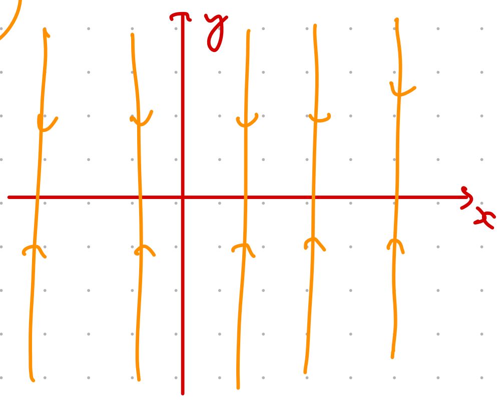
saddle

stable manifold is y axis

unstable manifold is x axis

[tracking back in time to where stable]

③ $\underline{a = 0}$



line of fixed points.

WHY NICE?
Eigenvectors are just the axes so nice

General System

$\dot{\underline{x}} = A\underline{x}$, try $\underline{x} = e^{\lambda t}\underline{v}$

$\dot{\underline{x}} = \lambda \cancel{e^{\lambda t}} \underline{v} = A \cancel{e^{\lambda t}} \underline{v}$

↖ eigenvalue of A.

can write down a general solution to this system.

$$\boxed{\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2}$$

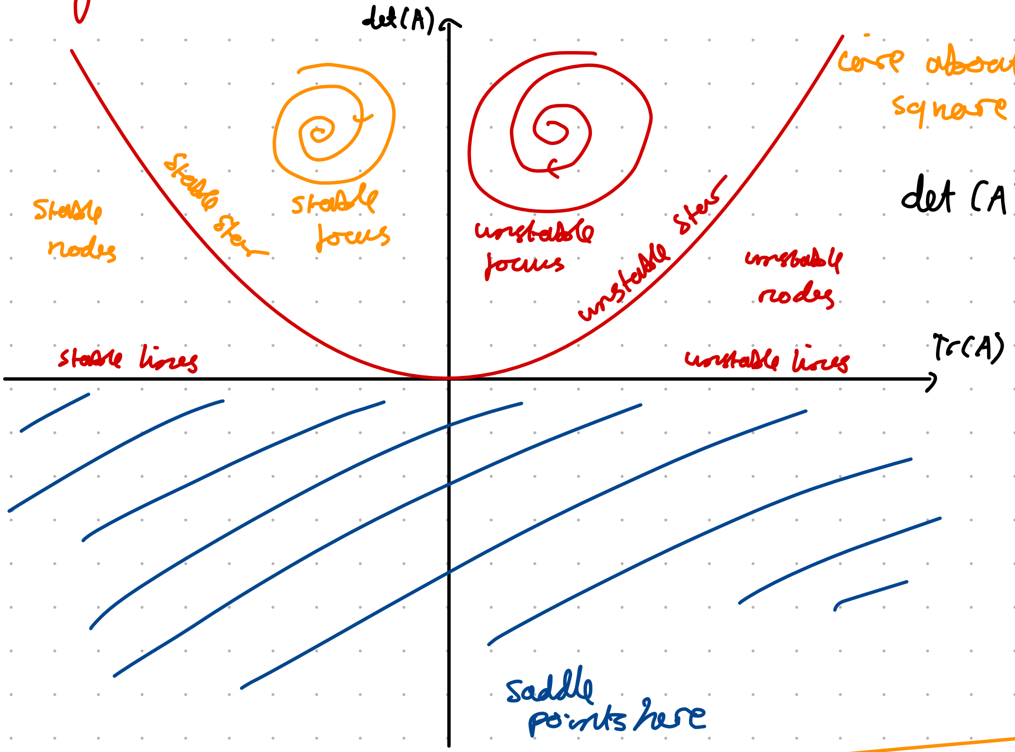
v_1, v_2 are eigenvectors of A

λ_1, λ_2 are eigenvalues.

λ_1, λ_2 are roots of $c_A(x) = 0$ $\Rightarrow$ $\lambda^2 - \underbrace{(a+d)}_{\text{trace } A}\lambda + \underbrace{(ad-bc)}_{\det(A)} = 0$

$$\Rightarrow \lambda_{1,2} = \frac{1}{2}\left(Tr(A) + \sqrt{(Tr(A))^2 - 4\det(A)}\right)$$

why bother with this? Easier to use tr & det ∵ can see relationships easier.



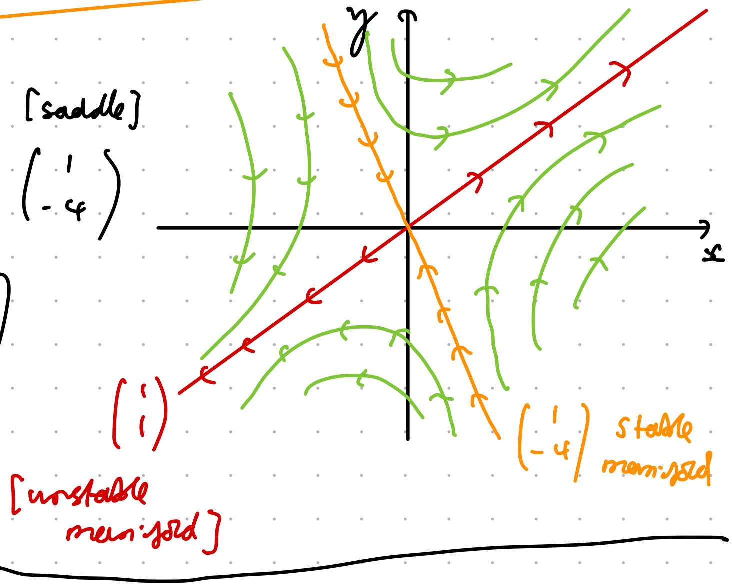
care about the sign under the square root.

$$\det(A) = \frac{1}{4}(Tr(A))^2$$

if $\det(A) < 0$, then λ_1, λ_2 always positive

Example: $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ $\lambda_{1,2} = 2, -3$ [saddle]

$v_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \& \begin{pmatrix} 1 \\ -4 \end{pmatrix}$



Non-Linear Systems

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$
or
$$\underline{\dot{x}} = f(\underline{x})$$

$x(0) = x_0$ so

$\phi(x_0, t)$

This is where we start & vary x_0 to get a family, again a flow

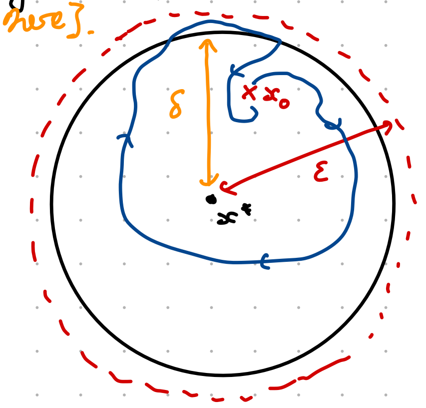
Def: A flow $\phi(x,t)$ is the solution at time t with inital value x at $t=0$.

Trajectories can't intersect or we loose uniqueness.

Def: A fixed point x^* is

① An attracting fixed point if all trajectories that start near x^* approach it as $t \longrightarrow \infty$

$\longrightarrow$ It's globally attracting if x^* attracts all trajectories.
[no matter where you start, will always end here].



② Lyapolov stable if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x(0) - x^*| < \delta$ then

$$|x(t) - x^*| < \varepsilon \quad \forall t > 0$$

where you start

fixed point

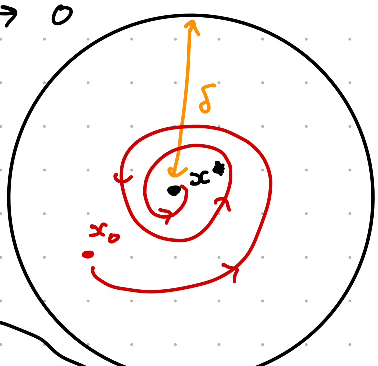
Start close - stay close stability.

③ Asymptotically stable if you're Lyapolov stable and $\exists \delta > 0$ s.t. if

$$|\underline{x}(0) - x^*| < \delta \Rightarrow |\underline{x}(t) - \underline{x}^*| \xrightarrow{t \to \infty} 0$$

Start within δ of fixed point.

will spiral into fixed point.



won't be asked for ε-δ based proof! Context for methods & theory, not exam.

Linearisation

$$\frac{dx}{dt} = ax + by = f(x,y)$$
$$\frac{dy}{dt} = cx + dy = g(x,y)$$

$$\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$$

"the jacobian"

$$\begin{pmatrix} a = \frac{\partial f}{\partial x} & b = \frac{\partial f}{\partial y} \\ c = \frac{\partial g}{\partial x} & d = \frac{\partial g}{\partial y} \end{pmatrix} \begin{matrix} \longleftarrow \dot{x} \\ \longleftarrow \dot{y} \end{matrix}$$

Proof: taylor expansion, in notes...

now take this:

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

fixed point (x^*, y^*)

disturbance
$u = x - x^*$
$v = y - y^*$

behaviour around fixed points is the same as linear system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$$

Thm: If $\dot{x} = f(x)$ has equilibrium x^* and a linearisation $\dot{x} = Ax$, then if A has no zero eigenvalues then the local stability of x^* is entirely determined by the linear system.

Population Models

(a) Predator Prey - 'Lotka Volterra'

Example: x prey y predators

(prey) $\frac{dx}{dt} = \alpha x - \beta x y$ (predators) $\frac{dy}{dt} = -\gamma y + \delta x y$

growth of prey unlimited.

how much predators eat

no food, no prey

infinite prey means predators keep growing

(b) Competing Species

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K_1} - \frac{\alpha_1 y}{K_1}\right) \qquad \frac{dy}{dt} = r_2 y\left(1 - \frac{y}{K_2} - \frac{\alpha_2 x}{K_2}\right)$$

proportion of max population size

$r_1 \boxed{\frac{x}{K_1}} \alpha_1 y$

limits population size of x

competing population

same but reflected.

E.g rabbits & sheep: (r, s)

not realistic, nice numbers!

$$\left.\begin{aligned}\dot{r} &= 3r\left(1 - \tfrac{r}{3} - \tfrac{2s}{3}\right)\\ \dot{s} &= 2s\left(1 - \tfrac{s}{2} - \tfrac{r}{2}\right)\end{aligned}\right] \Rightarrow \begin{aligned}\dot{r} &= r(3 - r - 2s) = f\\ \dot{s} &= s(2 - r - s) = g\end{aligned}$$

Fixed points: $\dot{r} = \dot{s} = 0 \Rightarrow r = 0$ or $s = 0$ ✓

① check $r = 0$: $\dot{s} = s(2 - s) = 0 \Rightarrow s = 0$ or $s = 2$

Fixed points at $(0,0)$ and $(0,2)$

② check $s = 0$: $\dot{r} = r(3 - r) = 0 \Rightarrow r = 0$ or $r = 3$

Fixed points at $(0,0)$ and $(3,0)$

③ $\left.\begin{aligned}3 - r - 2s &= 0\\ 2 - r - s &= 0\end{aligned}\right\} \longrightarrow$ fixed point at $(1,1)$

So fixed points are $(0,0), (0,2), (3,0), (1,1)$

want to linearise system to get eig

$$A = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{pmatrix} = \begin{pmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - r - 2s \end{pmatrix}$$

evaluate A at fixed points. Then eigenvalues etc.

Fixed point $(0,0)$: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\lambda_{1,2} = 3, 2$ $v_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_{1,2} > 0 \Rightarrow$ unstable node.

Fixed Point $(0,2)$ $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$ $\lambda_{1,2} = -2, -1$ $v_{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ #

just sheep, no rabbits (should stay here)

$\Rightarrow$ stable node $\because$ both λ negative

Fixed point (3,0)

just rabbits no sheep (should stay here)

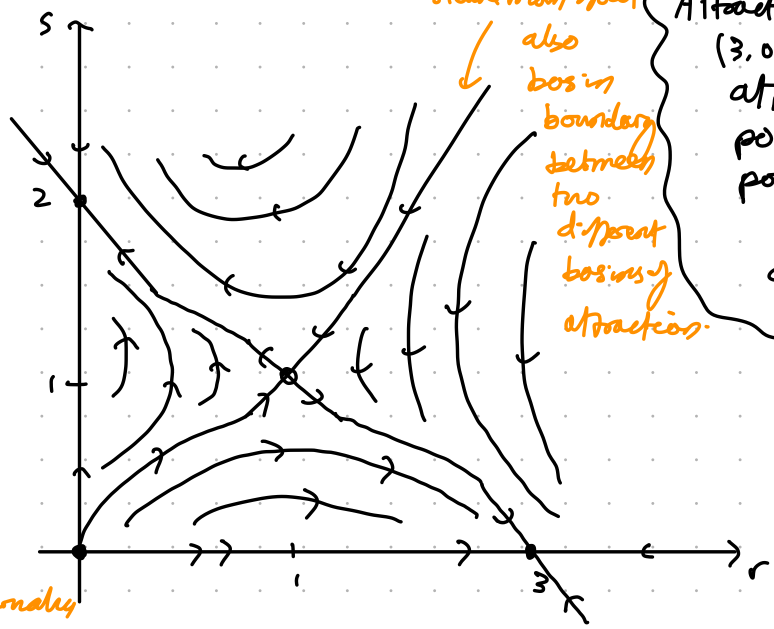
$A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ $\lambda_{1,2} = -3, -1$ $v_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

⇒ stable node ∵ both λ negative

Fixed point (1,1) $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ $\lambda_{1,2} = \underset{<0}{-\sqrt{2}-1}, \underset{>0}{\sqrt{2}-1}$ $v_{1,2} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

⇒ saddle ∵ one +ive, one -ive.

Now we draw a phase portrait



stable manifold also basin boundary between two different basins of attraction.

Attracting fixed points (e.g.) (3,0), (0,2) the basin of attraction of a fixed point is the set of possible x s.t. $x(t) \longrightarrow x^*$ as $t \longrightarrow \infty$

bigger eigenvalue for $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at (0,0) hence curve

Example: sheeps vs wolves

$\dot{w} = w(-a+bs)$ [wolves]

$\dot{s} = s(c-dw)$ [sheep]

jacobian matrix

$\dot{w} = \dot{s} = 0$ & fixed points at $(0,0)$, $(c/d, a/b)$

Linearise system: jacobian matrix $A = \begin{pmatrix} -a+bs & bw \\ -ds & c-dw \end{pmatrix}$

Sub in fixed points:

① (0,0) $A = \begin{pmatrix} -a & 0 \\ 0 & c \end{pmatrix}$ ⇒ $\lambda_{1,2} = -a, c$, $v_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

wolves sheep

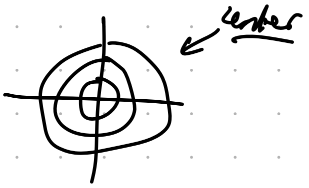
"no wolves so sheep grow exponentially"

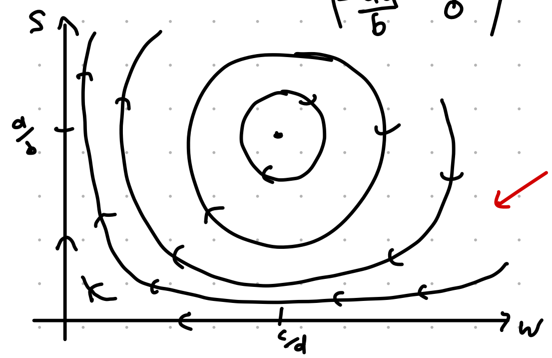
stable for wolves, unstable for sheep.

saddle

② $(c/d, a/b)$ $A = \begin{pmatrix} 0 & \frac{bc}{d} \\ -\frac{da}{b} & 0 \end{pmatrix}$ ⇒ $\lambda^2 + ac = 0$ ⇒ $\lambda = \pm i\sqrt{ac}$

So get circular orbits.





get these oscillations. lie on this trajectory.

Discrete Time Models

Discrete time can make more sense than continuous time. can get chaos...

1-D maps / difference equations / recurrence relations

$x_{n+1} = f(x_n)$, sequence $x_0, x_1, x_2, \ldots$ is the orbit starting from x_0.

$\boxed{\text{Def: } x^* \text{ is a fixed point if } f(x^*) = x^*}$

$x_n = x^* \Rightarrow x_{n+1} = f(x_n) = x^*$

Stability: write $\lambda = f'(x^*)$ where $'$ denotes $\frac{df}{dx}$

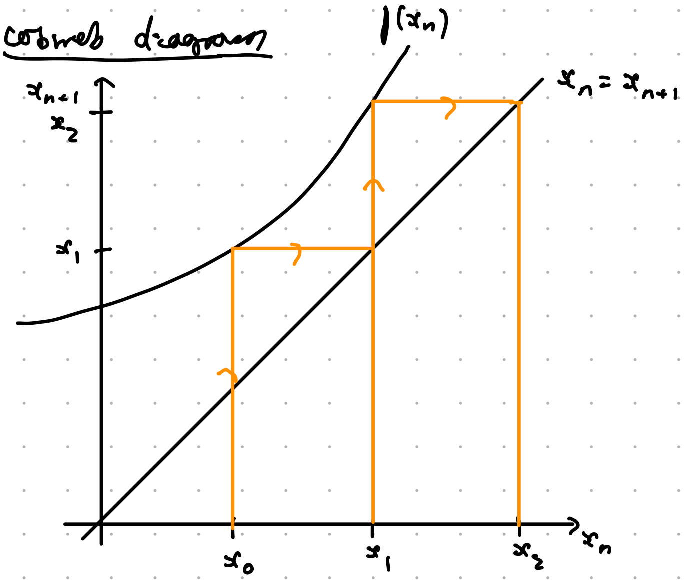
① If $|\lambda| < 1 \Rightarrow x^*$ is linearly stable

② If $|\lambda| > 1 \Rightarrow x^*$ is unstable

③ If $|\lambda| = 1 \Rightarrow$ marginal case - not sure...

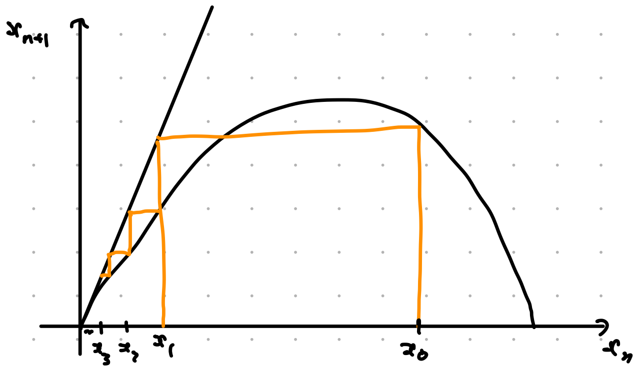
Draw a cobweb diagram to represent this.

Cobweb diagram



$x_{n+1} = f(x_n)$

Example $f(x_n) = \sin x_n$



Logistic Map

$x_{n+1} = r x_n (1 - x_n)$

normalized so x is a dimensionless measure

$x = 0$ min population

$x = 1$ max population

r is growth rate

$f(x) = rx(1-x) = x$ ← fixed points

If $\boxed{x = 0}$ fixed, if $x \neq 0$, $r(1-x) = 1 \Rightarrow \boxed{x = 1 - \frac{1}{r}}$ ← fixed points. $\boxed{r > 0}$

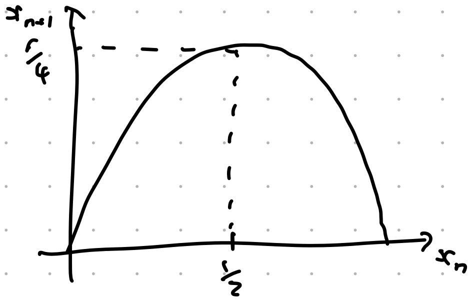
Note: this fixed point only exists for $r > 1$

$\Rightarrow$ bifurcation at $r = 1$. new fixed point...

$f(x) = rx(1-x)$ is a parabola with max at

$x_n = \frac{r}{4}$ [when $x_{n-1} = \frac{1}{2}$] $\Rightarrow$ care about $0 \leq r \leq 4$ so $0 \leq x \leq 1$

otherwise maximum makes no sense...



Look at stability of fixed points.

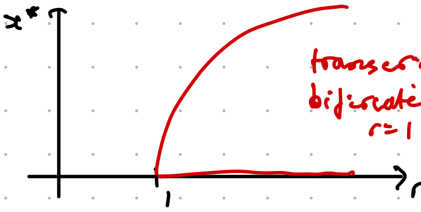
If $r < 1$: $x^* = 0$ is fixed point.

$f'(x) = r - 2rx$ $\quad |r| < 1$

$f'(0) = r < 1 \Rightarrow$ stable.

If $1 < r < 3$:, $x^* = 0$ & $x^* = 1 - \frac{1}{r}$

So $f'(0) = r > 1 \Rightarrow$ unstable

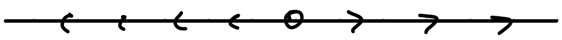


transcritical bifurcation $r=1$

$f'(1-\frac{1}{r}) = r - 2r + 2 = 2 - r \Rightarrow -1 < 2 - r < 1 \Rightarrow |2-r| < 1 \Rightarrow$ stable!

If $r > 3$:

DE



$x^* = 0$ unstable.

$x^* = 1 - \frac{1}{r}$ unstable

two fixed points & same type of stability.

Cyclic / Periodic orbits

$\exists n \in \mathbb{N}$ s.t. $f^n(x_0) = x_0$ & return to start point.

apply f n times

"n cycle / periodic orbit of period n"

E.g 2 cycle $\{p, q\}$ s.t.
$f(p) = q$, $f(q) = p$
$f^2(p) = p$.

$f^2(x) = r f(x)(1 - f(x)) = r^2 x(1-x)(1 - rx(1-x))$ ← quartic!

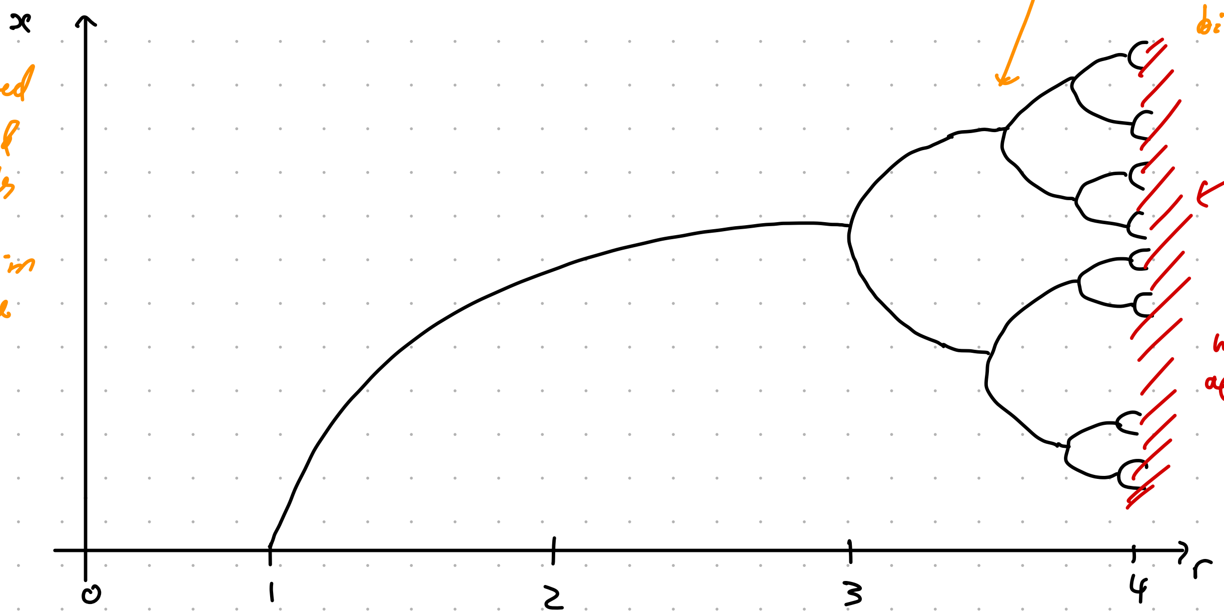
get harder & harder to find values... period 20 orbit is a huge polynomial. Too hard.

want $\underbrace{f^2(x) = x}_{\text{stay where you start!}}$ ← annoying to solve...

For $r > 3$, get one 2 cycle, but only r close to 3.

flip bifurcations / period doubling bifurcations

all fixed points & points that belong in a cycle



get chaos as you approach 4
⇓
weird stuff appearing...

Epidemiological Models

Toss up between complexity & ease of solving.

S – susceptable

E – exposed. [have the disease but not symptomatic]

I – infected [and infectious]

R – recovered/removed [might have some immunity]

V – vaccinated individuals

These are the 5 main compartments might add more based on the biology of the disease

The SIR model [basic start point]

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

transmission rate

$$\frac{dS}{dt} = -\beta S I$$

become infectious

more susceptible people

more infected people

$\frac{1}{\gamma}$ = duration of infectiousness

γ = recovery rate

$\gamma, \beta > 0$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

more into infection

proportion of the population

$$\boxed{S + I + R = 1}$$

everyone belongs to this

$$\frac{dR}{dt} = \gamma I$$

$S(0) > 0$, $I(0) > 0$ $R(0) = 0$

people are susceptable

this is a disease to start with...

For rate of growth of epidemic, look at $\frac{dI}{dt}$

for disease to be growing

$$\frac{dI}{dt} = \underbrace{I}_{\geq 0}(\underbrace{\beta S - \gamma}_{>0\,?})$$

If $S(0) > \beta/\gamma$ ← critical threshold The epidemic will grow...

transmission rate

$$\boxed{\frac{\beta}{\gamma} = r_0}$$

need to keep this smaller than one.

If $\gamma > \beta \Rightarrow \gamma/\beta > 1 \Rightarrow \boxed{\boxed{\beta/\gamma} < 1}$

$\frac{1}{\gamma}$ is duration of infectiousness

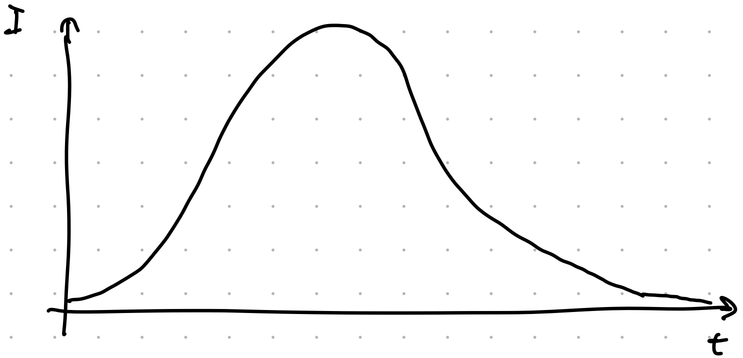
Def: the basic reproductive ratio r_0 is the average number of secondary cases arising from an average primary case in an entirely susceptable population.

If new disease, previously unseen, then:

$S(0) \simeq 1$

- $R_0 > 1$: pathogen can invade
- $R_0 < 1$: pathogen will die out.

In the real world, epidemics look like this.



$$\frac{dS}{dt} \div \frac{dR}{dt}$$

nice 1st order ODE

$$\frac{dS}{dR} = -\frac{\beta S \cancel{I}}{\gamma \cancel{I}} = -R_0 S$$

Integrate: $S(R) = Ae^{-R_0 R(t)}$

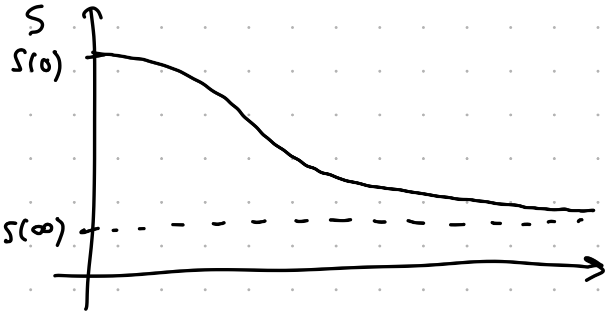
initial conditions,

$$S(t) = S(0)e^{-R_0 R(t)}$$

$[R(0)=0, \quad S(0)>0]$

So $S(t) = S(0)e^{-R_0 R(t)}$

$\Rightarrow$ $S(t)$ decreasing since $R(t)$ increasing



$S(\infty) = \lim_{t\to\infty} S(t)$, so always some susceptible individuals, must stop at some point.

$I(\infty) = 0$

$S+I+R=1 \Rightarrow S(\infty) + \cancel{I(\infty)} + R(\infty) = 1$

so $1 - R(\infty) - S(\infty)$

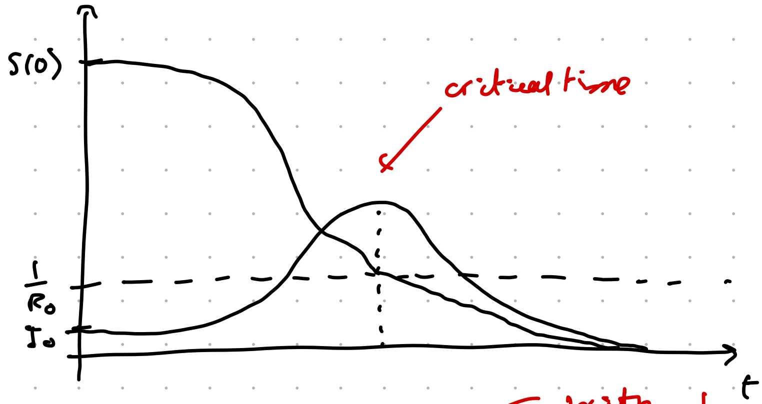
$$\boxed{1 - R(\infty) - S(0)e^{-R_0 R(\infty)} = 0}$$

final epidemic size, final number of people involved...

Why do we get epidemic burnout?

Assume $R_0 > 1$ [do get an epidemic], so $\frac{1}{R_0} < 1$

Then $\frac{dI}{dt} > 0$ if $S(0) > \frac{1}{R}$



critical time

Missing that recovered people can become susceptible again/ add births & deaths into the mix.

only fixed point is at 0 infected. No other fixed points.

with births/deaths

SIR with demography

death rate $\quad \frac{1}{\mu}$ is average lifespan

$$\frac{dS}{dt} = -\beta SI - \mu S + \mu$$

← birth rate × $\underbrace{\text{total population}}_{S+I+R=1}$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

Births & deaths balance to give constant population

When growing?

$$\frac{dI}{dt} = I(\beta S - \gamma - \mu) < 0 \text{ ? if } S < \frac{\mu+\gamma}{\beta}$$

So $R_0 = \frac{\beta}{\mu + \gamma}$ is our threshold for epidemic growth.

Fixed points

$I = 0$ or $S = \frac{\mu+\gamma}{\beta} = \frac{1}{R_0}$

① $\underline{I^* = 0}$: $-\mu S + \mu = 0 \Rightarrow S^* = 1$

Fixed point at $(1, 0, 0)$

[no infectious people, recovered eventually die off so only suceptables left]

② $S^* = \frac{1}{R_0}$: $\mu - \beta\left(\frac{\mu+\gamma}{\beta}\right) I - \mu\left(\frac{\mu+\gamma}{\beta}\right) = 0$

So $I^* = \frac{\mu}{\beta}(R_0 - 1)$ $\quad R^* = 1 - S^* - I^*$

Fixed point at $\frac{1}{R_0}, \frac{\mu}{\beta}(R_0 - 1), 1 - \frac{1}{R_0} - \frac{\mu}{\beta}(R_0 - 1))$ ← endemic equilibrium.

Find jacobian, $\frac{dS}{dt} = f(S, I, R)$, $\frac{dI}{dt} = g(S, I, R)$, $\frac{dR}{dt} = h(S, I, R)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial R} \\ \frac{\partial h}{\partial S} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial R} \end{pmatrix} = \begin{pmatrix} -\beta I - \mu & -\beta S & 0 \\ \beta I & \beta S - \mu - \gamma & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

← jacobian matrix. working out

Now, calculate eigenvalues...

Disease free equilibrium:

$$A = \begin{pmatrix} -\mu & \beta & 0 \\ 0 & \beta - \mu - \gamma & 0 \\ 0 & \gamma & \mu \end{pmatrix}$$

$\lambda_1 = -\mu < 0$

$\lambda_2 = 0$

$\lambda_3 = \beta - (\gamma + \mu)$

But, as $S + I + R = 1$, can reduce 3 equations to one,

Stable if $\lambda_3 < 0$, so $\beta < \mu + \gamma$, $\frac{\beta}{\mu+\gamma} = R_0 < 1$

$$\boxed{\text{note: } R(0) \neq R_0}$$

bit dumb!

Endemic equilibrium:

Require $R_0 > 1$, otherwise this won't exist.

Calculate determinant of $A - \lambda I$ before subbing in S & I

$$\det(A - \lambda I_3) = (-\mu - \lambda)((-\beta I - \mu - \lambda)(\beta S - (\mu + \gamma) - \lambda) + \beta I) \; ---$$

$\lambda_1 = -\mu$

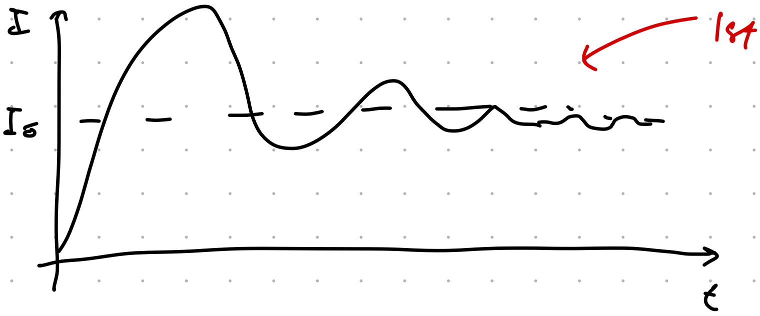
$$\lambda_{2,3} = -\frac{\mu R}{2} \pm \frac{1}{2}\sqrt{(\mu R)^2 - 4(\mu+\gamma)\mu\underbrace{(R_0-1)}_{>0}}$$

stable if $\lambda_{1,2,3} < 0$ [all less than zero]

① if $(\mu R)^2 < 4(\mu+\gamma)\mu(R_0-1)$ then get imaginary

∴ $\lambda_{2,3} = p \pm iq$ where $p = -\frac{\mu R_0}{2} < 0$

⇒ stable, real part negative with oscillations.

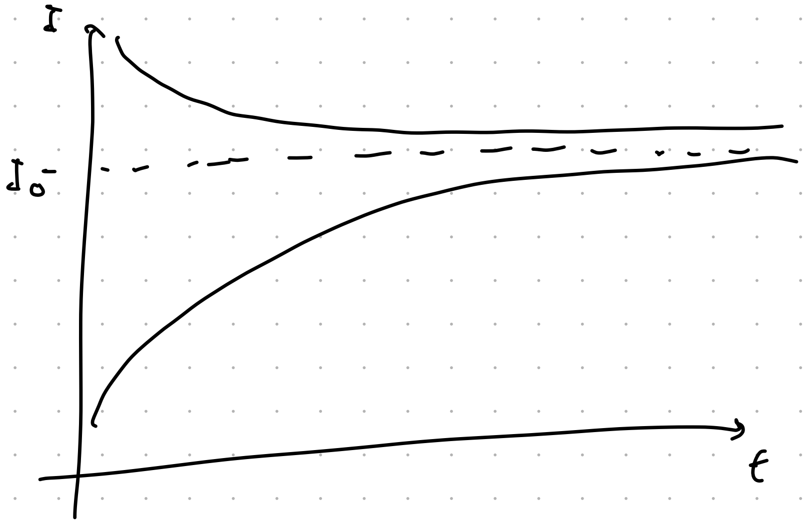


1st & 2nd waves damped oscillations.

② if $(\mu R)^2 > 4(\mu+\gamma)\mu(R_0-1) > 0$

$$\Rightarrow \frac{1}{2}\sqrt{(\mu R)^2 - 4(\mu+\gamma)\mu(R_0-1)} < \frac{1}{2}\mu R$$

so $\lambda_2, \lambda_3 < 0$, so stable, but no oscillations.



different cases for values of eigenvalues.

Average age for someone getting infected [mean]

Assume death rate $\mu << \beta$ [transmission $\frac{1}{\mu} = 65 \times 365$]

$\mu = \frac{1}{6 \cdot y}$ lifespan ↑ ↖ 1 year

Force of infection $= \beta I$

Average time spent in $S = \frac{1}{\beta I}$

At endemic equilibrium, $I^* = \frac{\mu}{\beta}(R_0 - 1)$

$a = \frac{1}{\beta I^*} = \frac{1}{\mu(R_0-1)}$ [mean age of infection]

SIS - model

No period of recovery. If you had a cold, can get it again 

$\Rightarrow$ only two compartments

$$\left.\begin{aligned} \frac{dS}{dt} &= -\beta S I + \gamma I \\ \frac{dI}{dt} &= \beta S I - \gamma I \end{aligned}\right] \qquad \frac{dS}{dt} = -\frac{dI}{dt} \qquad S + I = 1$$

$$S = 1 - I$$

$$\Rightarrow \frac{dI}{dt} = \beta I \left(1 - I - \gamma/\beta\right) = g(I)$$

$\leftarrow$ look at fixed points...

Stability: If $I^* = 0$, or $I^* = 1 - \gamma/\beta$ ... fixed points at

$(1, 0)$ and $\left(\frac{1}{R_0}, 1 - \frac{1}{R_0}\right)$

① $I^* = 0$,

$$g'(I) = \beta\left(1 - \frac{1}{R_0}\right) - 2\beta I$$

$$g'(0) = \beta\left(1 - \frac{1}{R_0}\right) \quad > 0 \quad \text{if } R_0 > 1 \quad [\text{unstable}]$$

② Endemic equilibrium $\quad < 0 \quad$ if $R_0 < 1 \quad$ [stable]

$$g'\left(1 - \frac{1}{R_0}\right) = -\beta\left(1 - \frac{1}{R_0}\right) < 0 \Rightarrow \text{stable}$$

SEIR model

Extra: $\frac{1}{\sigma}$ = time taken to become infectious [incubation period]

$$\frac{dS}{dt} = \mu - \beta S I - \mu S \qquad S + E + I + R = 1$$

deaths

$$\frac{dE}{dt} = \beta S I - \sigma E - \underbrace{\mu E}$$

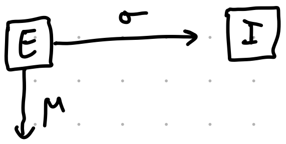
people becoming infectious

people can die before passing on disease

$\boxed{S} \quad \boxed{E} \quad \boxed{I} \quad \boxed{R}$

$$\frac{dI}{dt} = \sigma E - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$



How to find R_0: can calculate, or from definitions.

$$R_0 = \beta \times \underbrace{\left(\text{time spent infectious}\right)}_{\frac{1}{\mu+\gamma}} \times \underbrace{\left(\text{proportion that make it to being infectious}\right)}_{\frac{\sigma}{\mu+\sigma}}$$

transmission rate

can become infectious & then die

$$\boxed{R_0 = \frac{\beta\sigma}{(\mu+\gamma)(\mu+\sigma)}}$$

Disease Free Equilibrium: $(1,0,0,0)$

Endemic Equilibrium: $\frac{(\mu+\gamma)(\mu+\sigma)}{\beta\sigma} = S^* = \frac{1}{R_0}$

$$\left(\frac{1}{R_0},\ \frac{\mu(\mu+\gamma)}{\beta\sigma}(R_0-1),\ \frac{\mu}{\beta}(R_0-1),\ 1-S^*-I^*-E^*\right)$$

$\boxed{\text{Will not ask for } 4\times4 \text{ vector in an exam!}}$

$$A = \begin{pmatrix} -\beta I-\mu & 0 & -\beta S & 0 \\ \beta I & -(\mu+\sigma) & \beta S & 0 \\ 0 & 0 & -\mu-\gamma & 0 \\ 0 & 0 & \gamma & -\mu \end{pmatrix} \quad \swarrow \text{ jacobian}$$

Algebra ...

Eigenvalues: $\lambda_1 = -\mu < 0$,

$\lambda_2 \approx -(\sigma+\gamma) < 0$

$Re(\lambda_{3,4}) < 0$ if $R_0 < 1$

Stable if $R_0 < 1$

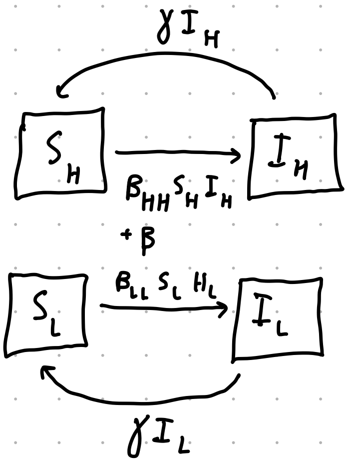
$\boxed{\text{Risk structure}}$ high risk & low risk ← only two types

$n_H + n_L = 1$, transmission rate β_{XY} = transmission rate from Y to X

← doing the infecting

Split into 4 compartments

High risk stay high risk & same w/ low



$$\frac{dS_H}{dt} = -\beta_{HH} S_H I_H - \beta_{HL} S_H I_L + \gamma I$$

$$\frac{dI_H}{dt} = \beta_{HH} S_H I_H + \beta_{HL} S_H I_L - \gamma I$$

$$\frac{dH_L}{dt} = \dots$$

$$\frac{dI_L}{dt} = \dots$$

so $S_H + I_H = n_H$ and $S_L + I_L = n_L$

who aquires infection from whom matrix [WAIFW matrix]

$$\begin{pmatrix} \beta_{HH} & \beta_{HL} \\ \beta_{LH} & \beta_{LL} \end{pmatrix} \begin{matrix} \longleftarrow \text{infection of } H \\ \longleftarrow \text{infection of } L \end{matrix}$$

Symmetric interactions...

can generally assume that $\beta_{HL} = \beta_{LH}$

Also $\beta_{HH} > \beta_{LL} > \beta_{HL}$

Example $\begin{pmatrix} 10 & 0.1 \\ 0.1 & 5 \end{pmatrix}$ ← very little transmission between high risk & low risk

$n_H = 0.2$, $n_L = 0.8$ $\gamma = 1$

↑ high risk is 20% ↑ low risk is 80% of population

$$R_0^H = \frac{\beta_{HH} n_H + \beta_{HL} n_L}{\gamma} = 2.08 > 1$$ ← infection takes off

$$R_0^L = \frac{\beta_{LL} n_L + \beta_{LH} n_H}{\gamma} = 0.82 < 1$$ ← infection dies off

But no! mixing between populations leads to a takeoff everywhere...

Disease free equilibrium

$S_L = n_L$, $S_H = n_H$

$$f(I_H, I_L) = \frac{dI_H}{dt} = (\beta_{HH} n_H - \gamma) I_H + \beta_{HL} \overset{S_H}{n_H} I_L$$

$$g(I_H, I_L) = \frac{dI_L}{dt} = (\beta_{LL} n_L - \gamma) I_L + \beta_{LH} n_L I_H$$

2×2 system of Differential Equations...

$$\text{Jacobian } A = \begin{pmatrix} \beta_{HH} n_H - \gamma & \beta_{HL} n_H \\ \beta_{LH} n_L & \beta_{LL} n_L - \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0.02 \\ 0.08 & -0.2 \end{pmatrix}$$

Find eigenvalues. $\lambda_1 = -0.2013$, $\lambda_2 = 1.0013$

$$\boxed{e^{\lambda_1 t} + e^{\lambda_2 t}}$$ ← for $t \approx 0$, larger term will dominate at the start...

$\lambda_2 > 0$ dominant eigenvalue ⇒ disease is going to invade.

R_0 didn't tell us what was going to happen... This does!

Assuming we know this categories. could have risk depending on age. You then get integral terms etc...

Vaccination

Aim: reduce # susceptibles (proportion), below the critical threshold.

so that $S < \boxed{\frac{1}{R_0}}$ ← found last time...

Assume: $S(0) + p = 1$ where p is the proportion vaccinated.

so want $S - p < \frac{1}{R_0} \Rightarrow p > 1 - \frac{1}{R_0}$

If $R_0 = 2$, need 50% vaccinated to ensure disease can't invade.

Example (Pediatric vaccination)

so vaccinate people at birth & scale...

$$\frac{dS}{dt} = \mu(1-p) - \beta S I - \mu S$$

birth rate

proportion not vaccinated

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \mu p + \gamma I - \mu R$$

those born & then vaccinated are immediately recovered!

change of variables

$S = S'(1-p)$

$I = I'(1-p)$

$R = R'(1-p) + p$

We change the variables to explore the system, setting $S = S'(1-p)$, $I = I'(1-p)$, $R = R'(1-p) + p$. Therefore:

$$(1-p)\frac{dS'}{dt} = \mu(1-p) - (\beta I'(1-p) + \mu)S'(1-p)$$
$$(1-p)\frac{dI'}{dt} = \beta S' I'(1-p)^2 - (\gamma + \mu)I'(1-p)$$
$$(1-p)\frac{dR'}{dt} = \gamma I'(1-p) + \mu p - \mu R'(1-p) - \mu p$$

which reduces to

$$\boxed{\begin{aligned}\frac{dS'}{dt} &= \mu - (\beta(1-p)I' + \mu)S' \\ \frac{dI'}{dt} &= \beta(1-p)S'I' - (\gamma+\mu)I' \\ \frac{dR'}{dt} &= \gamma I' - \mu R'\end{aligned}}$$

nice

[SIR model with $\beta' = \beta(1-p)$]

Require $R_0 < 1 \Rightarrow (1-p)R_0 < 1 \Rightarrow p > \boxed{1 - \frac{1}{R}} = p_c$

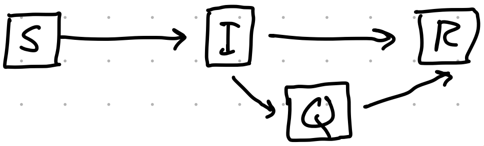
measles needs 97% vaccinated so depends on public behaviour...

critical number of vaccinations so that disease doesn't break out...

Isolation [Q for quarantine]

Assume: detection rate d

time spent in $Q = \frac{1}{\tau}$



Include demography... births - infections - deaths.

$$\frac{dS}{dt} = \mu - \beta I S - \mu S$$

$$\frac{dI}{dt} = \beta I S - dI - \gamma I - \mu I$$

$$\frac{dQ}{dt} = dI - \tau Q (- \mu Q)$$

no deaths in quarantine

$$\frac{dR}{dt} = \gamma I + \tau Q - \mu R$$

$R_0 = \frac{\beta}{d+\gamma+\mu} = 1 \Rightarrow \boxed{d = \beta - \mu - \gamma}$ ← critical rate to ensure infection doesn't take off...

Setting all $\frac{d?}{dt} = 0$, $\boxed{\text{Equilibrium}}$ at

$$I^* = 0 \quad \text{or} \quad S^* = \frac{1}{R_0} = \frac{d+\gamma+\mu}{\beta}$$

Disease free: $(1, 0, 0, 0)$

only exists if

Endemic eq: $\left(\frac{1}{R_0}, \frac{\mu}{\beta}(R_0-1), \frac{d\mu}{\tau\beta}(R_0-1), \frac{R_0-1}{\beta}(\gamma+d)\right)$ $\boxed{\text{if } R_0 > 1}$

susceptibles

quarantine

Big assumption: no limitations on ability to quarantine people.

What if we had that $Q = \boxed{Q_c}$ ← some upper bound on quarantine.

$dI \geq \tau Q_c$

not dependent on # infectious people...

$\frac{dS}{dt}$ stays same. $\frac{dI}{dt} = \beta SI - (\gamma+\mu)I - \boxed{\tau Q_c}$

$\frac{dQ}{dt} = \tau Q_c - \tau Q_c = 0$ so $Q = Q_c$ constant!

so $\frac{dR}{dt} = \tau Q_c + \gamma I - \mu R$

At threshold, jump to a different model so get weird discontinuities

$R_0^c = \frac{\beta}{\gamma + \frac{\tau Q_c}{I^*} + \mu} > \boxed{R_0}$ ← before we hit critical threshold

If $\boxed{Q < Q_c}$ and $R_0 > 1$, we get ① a stable endemic equilibrium
② unstable disease free equilibrium.

If $\boxed{Q = Q_c} \Rightarrow R_0^c > R_0 > 1$, set all equal to zero, get quadratic in $I = 0$, get one low equilibrium & one higher

This means

① I^*_{low}

② I^*_{high}

Discrete Time Models

$S + I + R = N$ ← population size . Assume no demography (no births & deaths)

Assume a timestep $n = \Delta t$

At each timestep two possible events

① $S \longrightarrow S-1$, $I \longrightarrow I+1$ [infection]

② $I \longrightarrow I-1$ $R \longrightarrow R+1$ [recovery]

Write down difference equations

$$\boxed{\text{Euler's method: if } \frac{dx}{dt} = f(x), \text{ then } x_{n+1} = x_n + f(x)\Delta t}$$

apply ↓↓

$$f(S_n, I_n, R_n) = S_{n+1} = S_n - \frac{\beta S_n I_n}{N}\Delta t + \mu \Delta t N - \mu \Delta t S_n$$

$$g(S_n, I_n, R_n) = I_{n+1} = I_n + \frac{\beta S_n I_n}{N}\Delta t - (\gamma + \mu)\Delta t I_n$$

$$h(S_n, I_n, R_n) = R_{n+1} = R_n + \gamma \Delta t I_n - \mu \Delta t R_n$$

Susceptibles at timestep n+1 is ______

Fixed points: $S_{n+1} = S_n$, $I_{n+1} = I_n$, $R_{n+1} = R_n$

Disease free Equilibrium: $S_n^* = N$, $I_n^*, R_n^* = 0$

Endemic Equilibrium: $S_n^* = \frac{N}{R_0}$, $I_n^* = \frac{\mu N}{\beta}(R_0 - 1)$, $R_n^* = \frac{\gamma N}{\beta}(R_0 - 1)$

Now want to consider stability, so construct a master equation.

$P_I(t)$ = probability of I infectious individuals at time t

$\{P_I(t)\}_{I=0}^{N}$ ← this is our distribution...

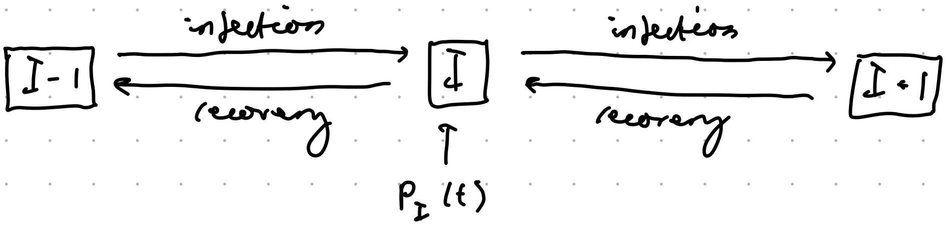
Example SIS model w/ no demography (no births/deaths)

Two events:

① $S \longrightarrow S-1$, $I \longrightarrow I+1$ infection rate: $\frac{\beta S I}{N}$

② $S \longrightarrow S+1$, $I \longrightarrow I-1$ recovery rate: γI

4 events / interactions that impact probability that I ppl infectious



① situation w/ I infected, has recovery at rate γI additional susceptible

② situation w/ I-1 ppl infected. has infection at rate $\frac{\beta(S+1)(I-1)}{N}$

$$= \frac{\beta(N-I+1)(I-1)}{N}$$

③ situation w/ I ppl infected, has a new infection $= \frac{\beta S I}{N} = \frac{\beta(N-1)I}{N}$

④ situation w/ I+1 ppl infected, has recovery at rate $\gamma(I+1)$

master equation: P_I[rate of event A] = prob of being in state I and event A occurs.

$$\frac{dP_I}{dt} = -P_I[\gamma I] - P_I\left[\frac{\beta(N-1)I}{N}\right] + P_{I-1}\left[\frac{\beta(N-I+1)(I-1)}{N}\right] + P_{I+1}[\gamma(I+1)]$$

We want the distribution of all possible scenarios & their probability

$$\frac{dP_I}{dt} = 0$$

Balance occurs when recovery from I+1 balances infections from I

$$P_{I+1}^*[\gamma(I+1)] = P_I^*\left[\frac{\beta(N-I)I}{N}\right]$$

I to I-1

Rearranging

$$P_{I+1}^* = P_I^* \frac{\beta(N-I)I}{N\gamma(I+1)} = P_{I-1}^*\left(\frac{\beta(N-I)I}{N\gamma(I+1)}\right)\left(\frac{\beta(N-I+1)(I-1)}{N\gamma I}\right)$$

keep going & iterating

$$P_{I+1}^* = P_1^* \prod_{j=1}^{I}\left(\frac{\beta(N-j)j}{N\gamma(j+1)}\right)$$

So can work out equilibrium distribution $\{P_i\}_{i=1}^{N}$ if we know P_1^*

Simplify:

$$P_I^* = P_1 \frac{(N-1)!}{(N-I)!\,I}\left(\frac{\beta}{\gamma N}\right)^{I-1}$$

Need P_1.

Extinction rate: only happens when $I=1 \longrightarrow I=0$

rate = γP_1 ← probability only one infectious individual

$$\underbrace{\sum_{I=1}^{N} P_I}_{=1} = P_1 \sum_{I=1}^{N} \frac{(N-1)!}{(N-I)!\,I}\left(\underbrace{\frac{\beta}{\gamma N}}_{R_0}\right)^{I-1}$$

Rearrange:

$$P_1 = \left(\sum_{I=1}^{N} \frac{(N-1)!}{(N-I)!\,I} R_0^{I-1}\right)^{-1}$$

can now calculate P_1 so know the entire distribution of all possible outcomes.

Extinction rate = γP_1 is rate of extinction for SIS model without demography

master equation method is a powerful way to get any possible distributions.

Better than DEs ∵ you can say exactly when you reach extinction. Before, dies out at infinity. Now can say when it ends exactly.

Chapter 3 - Modelling molecular networks

We now consider within-host biological models. Now into the micro scale. Chemical reactions converting one molecule into another.

Simple example

Assume a molecule A is produced & degraded at a constant rate such that

$$0 \underset{k_1}{\overset{k_2}{\rightleftharpoons}} A$$

k_1 = production rate

k_2 = degradation rate

Production of A assumed constant over time.

$$\frac{d[A]}{dt} = k_1 - k_2[A]$$

$[A]$ = concentration of molecule A

Fixed points occur when $\frac{d[A]}{dt} = 0$. Assume $k_1, k_2 > 0$

So equilibrium when $[A] = \frac{k_1}{k_2}$

So $\frac{d[A]}{dt} > 0$ when $[A] < \frac{k_1}{k_2}$

$\frac{d[A]}{dt} < 0$ when $[A] > \frac{k_1}{k_2}$

} fixed point is globally stable...

Section 3.1 - Autocatalysis

System w/ a molecule X which induces its own production with the addition of a molecule A.

$$A + X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 2X$$

Use Law of mass action: states that the rate of a reaction is proportional to

Assume a surplus on A ⇒ A's concentration is constant.

This gives us

$$f([X]) = \frac{d[X]}{dt} = k_1[A][X] - k_2[X][X]$$

$$= [X](k - k_2[X]) \quad \text{for } k = k_1[A]$$

Fixed points occur when $\frac{d[X]}{dt} = 0 \Rightarrow [x] = 0$ or $[x] = \frac{k}{k_2}$

Differentiate: $$\frac{\partial f}{\partial [X]} = k - 2k_2[x]$$

when $[X] = 0 \Rightarrow \frac{\partial f}{\partial [x]} > 0 \Rightarrow$ unstable

. . .

Example: Two molecules X & Y form a complex. Induce their own production in presence of A, B. ~~X_4~~ The system can be described as

$$A + X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 2X$$

$$B + Y \underset{k_5}{\overset{k_4}{\rightleftharpoons}} 2X$$

$$X + Y \xrightarrow{k_3} C$$

$$f([x],[Y]) = \frac{d[x]}{dt} = k_1[A][X] - k_2[A][X] - k_3[X][Y]$$

big small

$$K_1 = k_1[X] \qquad = [X][K_1 - k_2[X] - k_3[X]]$$

Similarly: $g([X],[Y]) = \frac{d[Y]}{dt} = [Y](K_2 - k_5[Y] - k_3[X])$

Equilibrium when $\frac{d[x]}{dt} = \frac{d[Y]}{dt} = 0$

Then find fixed points... (see notes I ~~too~~ tired...)

To find stability, compute the jacobian J

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\delta g}{\delta y} \end{pmatrix}' =$$

Eigenvalues at (0, 0)

Stable / unstable direction sum

Law of mass action: rate of chemical reaction. , —

$$s_1 S_1 + s_2 S_2 \longrightarrow p_1 P + p_2 P_2$$

Direct prop -

Example

Species A, B, form a complex C.

forward reaction rate

$$A + B \underset{k_b}{\overset{k_f}{\rightleftharpoons}} C$$

backward rate