Fluid Dynamics Summary

Chapter 2 - Mathematical Modelling of Fluid Flow

① Dynamic Viscosity μ (Newtonian viscosity), Resistance of a fluid to shear.
Inviscid $\Rightarrow$ $\mu = 0$ $\quad [\mu] = \frac{Ns}{m^2} = \frac{MLT^{-2}T}{L^2} = \frac{M}{LT}$
↳ $\nu = \frac{\mu}{\rho}$ is kinematic

$F(X, t_0) = X$ ← initial position

② $\underbrace{\text{Eulerian}}_{x,\ \text{fixed}}$, $\underbrace{\text{Lagrangian}}_{X \text{ in flow}}$. $\quad x = F(X, t)$, $\left.\frac{\partial F}{\partial t}\right|_{X \text{ fixed}} = u(F(X,t), t)$

③ Material Derivative. $\frac{D}{Dt} = \frac{\partial}{\partial t} + \overbrace{u \cdot \nabla}^{\text{convective rate of change}}$ $\quad$ e.g. $a = \frac{Du}{Dt}$

BE CAREFUL TO ADAPT TO COORDINATE SYSTEM

b some property

④ Reynold's Transport Thm $\quad \frac{D}{Dt}\int_{\Omega} b\, d\Omega = \int_{\Omega} \frac{\partial b}{\partial t} d\Omega + \int_{\partial\Omega} b(u\cdot\hat{n})\, dA$

⑤ Flow visualisations

ⓐ Particle Paths, where does a Lagrangian fluid element go?

$$\frac{\partial F}{\partial t} = u \qquad w/ \qquad F(X, 0) = X_0$$

ⓑ Streamlines, curve everywhere to eulerian velocity vector $\frac{dx}{\partial s} = u$ fixed t

$\boxed{\text{In a steady flow, 'particle paths' = 'streamlines'}}$

⑥ Taylor series on velocity field

$$u_i(x_j + \delta x_j, t) = u_i(x_j, t) + \frac{\partial u_i}{\partial x_j}\delta x_j = u_i(x_j) + r_{ij}\,\delta x_j + e_{ij}\,\delta x_j$$

$$\underbrace{r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{anti-symmetric rate of rotation tensor}} \qquad \underbrace{e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric rate of strain tensor}} \qquad \omega = \nabla \times u$$

⑦ Conservation laws $\quad \frac{D}{Dt}\int_{\Omega} b\, d\Omega = \underbrace{\int_{\Omega} q^B\, d\Omega}_{\text{volumetric source term}} + \underbrace{\int_{\partial\Omega} h^B\, dA}_{\text{surface source terms}}$ $\quad \int_{\Omega} \frac{\partial T^B_{\cdot j}}{\partial x_j} d\Omega$

→ mass: $b = \rho$, $q^B = 0$, $T^B = 0$. Assuming incompressible $\Rightarrow$ $\nabla \cdot u = 0$

→ momentum: $b = \rho u$, $q^B = \rho g$, $T^B = T$ $\Rightarrow$ $\rho \frac{Du}{Dt} = \rho g_i + \frac{\partial T_{ij}}{\partial x_j}$

Cauchy stress tensor

→ Angular momentum: $b = x \times \rho u$, $q^B = x \times \rho g$, $T^B = x \times T$ $\Rightarrow$ $T = T^T$ [Symmetric]

⑧ Ideal / Inviscid fluid: $\quad T_{ij} = -p\,\delta_{ij}$ [No friction between fluid elements] pressure acts inwards normally

↳ Euler equation: $\rho \frac{Du}{Dt} = -\nabla p + \rho g$

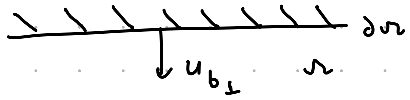
⑨ A Newtonian Fluid: $\quad T_{ij} = -p\,\delta_{ij} + \sigma_{ij} = -p\,\delta_{ij} + \overbrace{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}^{\text{deviatoric stress tensor}}$

Note, sub into momentum balance to get incompressible Navier Stokes

⑩ Boundary conditions

(Full Navier Stokes)
Viscous Fluids : Fluid velocity at boundary is the boundary velocity
[Honey sticks to wall] $u(x,t) = U_b(x,t) \quad \forall x \in \partial\Omega$ [No slip]

(no $\nabla^2 u$ term)
Inviscid Fluids : can only enforce impenetrability of boundary. Velocity normal
[water can flow] to boundary must match boundary. No condition parallel
$\therefore$ you can slip along boundary



$u_\perp(x,t) = U_{b\perp}(x,t) \quad \forall x \in \partial\Omega$

Chapter 3 - 1d Flows

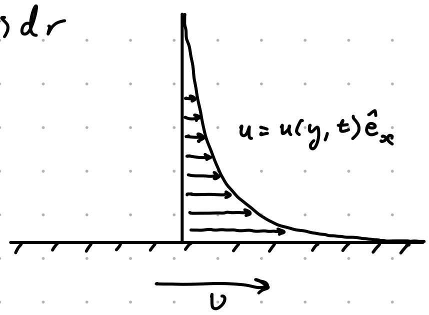
⑪ Flow in a pipe

ⓐ Axial force exerted by fluid inside of cylinder on fluid outside: $F_z = \underbrace{2\pi r}_{\text{Area}} \underbrace{\sigma_{rz}}_{\text{Stress}}$

ⓑ Volumetric flow rate $Q = \int_\Omega w(r)\,d\Omega = 2\pi \int_0^R r\,w(r)\,dr$

⑫ Stokes' First Problem

$\nabla \cdot u = 0$
$\rho\frac{\partial u}{\partial t} + \left(\begin{smallmatrix}u\\0\\0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}\partial_x\\\partial_y\\\partial_z\end{smallmatrix}\right)\left(\begin{smallmatrix}u\\0\\0\end{smallmatrix}\right) = -\nabla p + \mu\nabla^2 u + \rho g$



with $u = u(y,t)\hat{e}_x$, Navier Stokes simplifies to

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad 0 = -\frac{\partial p}{\partial y}$$

As $u \to 0$ as $y \to \infty$, $p(x,t)$ only
so $\frac{\partial p}{\partial x} = 0$ as $y \to \infty \Rightarrow p(t)$ so

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \quad (1) \\ &u = U \quad \text{at } y = 0 \\ &u = 0 \quad \text{as } y \to \infty \\ &u = 0 \quad \text{at } t = 0 \end{aligned}}$$

Equations have scaling symmetry! TRANSFORM $t = \alpha\tilde{t},\ y = \beta\tilde{y}$, $u = \gamma\tilde{u}$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tilde{u}}\cdot\frac{\partial \tilde{u}}{\partial \tilde{t}}\cdot\frac{\partial \tilde{t}}{\partial t} = \gamma\frac{\partial \tilde{u}}{\partial \tilde{t}}\frac{1}{\alpha}$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial \tilde{u}}\cdot\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\left(\frac{\partial \tilde{y}}{\partial y}\right)^2 = \frac{\gamma}{\beta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$

$\frac{\cancel{\gamma}}{\alpha}\frac{\partial \tilde{u}}{\partial \tilde{t}} = \nu\frac{\cancel{\gamma}}{\beta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$

$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \nu\frac{\alpha}{\beta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$

$\tilde{u}(\tilde{y}=0) = \frac{U}{\gamma}$
$\tilde{u}(\tilde{y}\to 0) = 0$
$\tilde{u}(\tilde{t}=0) = 0$

$$\boxed{\begin{aligned} &\text{Rescaling yields} \\ &\frac{\partial \tilde{u}}{\partial \tilde{t}} = \nu\frac{\alpha}{\beta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \quad \tilde{u}(\tilde{y}=0,\tilde{t}) = \frac{U}{\gamma} \\ &(2) \qquad\qquad \tilde{u}(\tilde{y}\to\infty,\tilde{t}) = 0 \\ &\qquad\qquad\quad\ \tilde{u}(\tilde{y},\tilde{t}=0) = 0 \end{aligned}}$$

If $\gamma = 1$, $\alpha = \beta^2$
Then (1) = (2) so
$u(y,t) = \tilde{u}(\tilde{y},\tilde{t})$
$= u\left(\frac{y}{\beta}, \frac{t}{\beta^2}\right)$
β free so pick $\beta^2 = t$
Then $u(y,t) = u\left(\frac{y}{\sqrt{t}}, 1\right)$

This suggests the functional form for the similarity variable

OR $\left.\begin{aligned}\frac{\partial u}{\partial t} &\sim \frac{u}{t} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{u}{y^2}\end{aligned}\right]$ $\frac{\cancel{u}}{t} \sim \nu\frac{\cancel{u}}{y^2}$, $U \sim u$ $\Rightarrow y \sim \sqrt{\nu t}$

U gives dimensions
$f(\xi)$ dimensionless

$\underbrace{\xi = \frac{y}{\delta(t)}}_{\text{dimensionless!}}$ where $\underbrace{\delta(t) = 2\sqrt{\nu t}}_{\text{Scale for } y}$

THEN $u(y,t) = U f(\xi)$
sub in f solve for f

Chapter 4 - Dimensional Analysis

If drag $D = f(\rho, \mu, R, U)$, scale $\tilde{l} = \alpha l$, $\tilde{t} = \beta t$, $\tilde{m} = \gamma m$
will also have $\tilde{D} = f(\tilde{\rho}, \tilde{\mu}, \tilde{R}, \tilde{U}) \Rightarrow$ so pick α, β, γ

Chapter 5 - Bernoulli Equation

(13) Hydrostatics: $u=0 \Rightarrow \nabla p = \rho g \Rightarrow p = p_0 + \rho g \cdot x$

$$u \times \omega = u \times (\nabla \times u) = \nabla\left(\tfrac{1}{2}|u|^2\right) - (u\cdot\nabla)u$$

(14) Bernoulli Equation

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \mu\nabla^2 u + \rho g$$ [just Navier Stokes]

$$\rho\frac{\partial u}{\partial t} + \rho\nabla\left(\tfrac{1}{2}|u|^2\right) - \rho\, u\times\omega = -\nabla p + \mu\nabla^2 u + \rho g$$

$$\left[\nabla(g\cdot x) = \nabla(g_1 x + g_2 y + g_3 z) = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\right]$$

$$\rho\frac{\partial u}{\partial t} + \nabla\Big(\underbrace{\rho\tfrac{1}{2}|u|^2 + p - \rho g\cdot x}_{\tilde{B}}\Big) = \mu\nabla^2 u + \rho\, u\times\omega$$

(a) Steady incompressible inviscid flow: $\nabla\tilde{B} = \rho\, u\times\omega$

Note: $u\cdot\nabla B = \rho u\cdot(u\times\omega) = 0$, $\omega\cdot\nabla B = \rho\omega\cdot(u\times\omega) = 0$] $\tilde{B}$ constant along streamlines / vortex lines

(b) Unsteady incompressible potential flow: $u = \nabla\phi$ — $\hat{B}$ could vary w/ time

Laplace eqn: $\nabla\cdot u = 0 \Rightarrow \nabla\cdot(\nabla\phi) = \nabla^2\phi = 0$

vorticity: $\omega = \nabla\times u = \nabla\times(\nabla\phi) = 0$

RHS: $\nabla^2 u = \nabla\cdot(\nabla u) = 0$, $\omega = 0$

LHS: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\nabla\phi) = \nabla\frac{\partial\phi}{\partial t}$

$$\nabla\hat{B} = 0$$

$$\hat{B} = \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|u|^2 + p - \rho g\cdot x$$

Chapter 6 - Potential Flow

(15) Conditions: $\underbrace{\nabla\cdot u = 0}_{\text{incompressible}}$, $\underbrace{\nabla\times u = 0}_{\text{irrotational}} \iff u = \nabla\phi$

$\nabla\phi\cdot\hat{n} = 0$

Note: no normal flow B.C. $\Rightarrow u\cdot\hat{n} = 0$ [on solid walls]

No slip condition cannot be satisfied by potential flow past solid objects

(16) 2d: $w = \phi + i\psi \Rightarrow \frac{dw}{dz} \overset{\text{complex analysis}}{=} \frac{\partial w}{\partial x} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv$

$$\nabla\times u = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow u = \frac{\partial\phi}{\partial x},\ v = \frac{\partial\phi}{\partial y}$$

$$\nabla\cdot u = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}$$

$\nabla\phi\cdot\nabla\psi = 0$ too...

(17) Streamlines: $\underbrace{\text{contours of } \psi \text{ are streamlines}}_{\psi \text{ constant along stream lines}}$! $\nabla\psi\cdot u = \frac{\partial\psi}{\partial x}u + \frac{\partial\psi}{\partial y}v = 0$

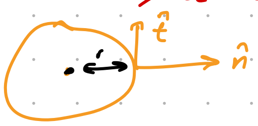
(18) Elementary Flows

- Uniform flow: $w(z) = Ue^{i\alpha}z$
- Point source: $w(z) = \frac{Q}{2\pi}\log z$
- Point vortex: $w(z) = \frac{-i\Gamma}{2\pi}\log z$
- Point dipole: $w(z) = \frac{D}{2\pi z}$
- Higher multipoles: $w(z) = \frac{a_k}{2\pi z^k}$

Power Series

$$w(z) = \underbrace{Ue^{i\alpha}z + \frac{Q - i\Gamma}{2\pi}\log z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots}_{\text{can get any flow you want!}}$$

note, pt source $z = re^{i\theta} \rightarrow u = \frac{Q}{2\pi r}\hat{e}_r$, $Q = \int_C u\cdot\hat{n}\,ds = \int_0^{2\pi}\frac{Q}{2\pi r}\hat{e}_r\cdot\hat{e}_r\, r\,d\theta = Q$



$? = \int_C u\cdot\hat{t}\,ds$

Integrate around a circle of radius r

(19) Immersed bodies

- Rankine half body: $w(z) = Uz + \frac{Q}{2\pi}\log z$
- Rankine oval: $w(z) = Uz + \frac{Q}{2\pi}\left(\log(z-1) + \log(z+1)\right)$
- Circle: $w(z) = U\left(z + \frac{a^2}{z}\right)$

(20) Force on Immersed bodies: (force per unit length out of the plane of paper)

$$F = \int_C -p\hat{n}\,ds \qquad p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|u|^2\right) + \rho g\cdot x$$

C is the bding closed streamline defining the body [From Bernoulli]

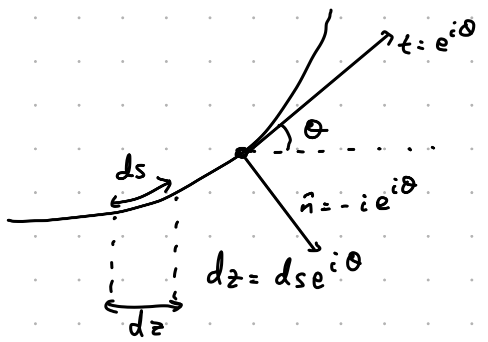
complex force

(21) Force in steady potential flow: $F = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \longleftrightarrow F = F_x + iF_y$

$u = |u|e^{i\theta}$, $w'(z) = u - iv = |u|e^{-i\theta}$ so $|u|^2 = (w'(z))^2 e^{2i\theta}$

complex conjugate

$$\bar{F} = F_x - iF_y = \int_C -p\bar{n}\,ds = \int_C \frac{\rho}{2}|u|^2\,\bar{n}\,ds \qquad n = -ie^{i\theta}$$



$$= \frac{\rho}{2}\int_C (w'(z))^2 e^{2i\theta}\cdot \underbrace{ie^{-i\theta}}_{\bar{n}}\,\underbrace{e^{-i\theta}dz}_{ds}$$

$$= \frac{i\rho}{2}\int_C (w'(z))^2\,dz$$

$$= \frac{i\rho}{2}2\pi i\,\mathrm{Res}\left((w'(z))^2, 0\right)$$

$$\boxed{\begin{aligned} w &= Uz + \frac{Q - i\Gamma}{2\pi}\log z + \ldots \\ w' &= U + \frac{Q - i\Gamma}{2\pi z} + \ldots \\ (w')^2 &= \frac{2U(Q - i\Gamma)}{2\pi z} + \ldots \end{aligned}}$$

conservation of mass: $Q = 0$

$$= \frac{i\rho}{2}2\pi i\cdot\frac{2U(Q - i\Gamma)}{2\pi}$$

[Cauchy's thm for contour integral]

$F_x = 0$, $F_y = -\rho U\Gamma$

$$= -\rho U(Q - i\Gamma)$$

$\Rightarrow$ Lift force $= \rho U\Gamma$

(22) Added mass from unsteady term: $\rho\frac{\partial\phi}{\partial t} \Rightarrow F = \int_C \rho\frac{\partial\phi}{\partial t}\hat{n}\,ds$

E.g. circle

$w(z) = U\left(z + \frac{a^2}{z}\right)$

real part of w

$w = \phi + i\psi \Rightarrow \phi = \overbrace{Ur\cos\theta + \frac{Ua^2}{r}\cos\theta}^{} = U\cos\theta\left(r + \frac{a^2}{r}\right)$

on $r = a \Rightarrow \phi = 2Ua\cos\theta$ [boundary c]

$$\therefore\quad F = \int_C -p\hat{n}\,ds = \int_0^{2\pi} \rho\,2U'a\cos\theta\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}a\,d\theta = \begin{pmatrix}\rho\pi a^2\frac{dU}{dt} \\ 0\end{pmatrix}$$

(23) Flow around an airfoil: conformal map $z = f(\zeta) = (1+\varepsilon)\zeta + (1-\varepsilon)\frac{a^2}{\zeta}$. $z = ae^{i\theta} \rightarrow$ plate.

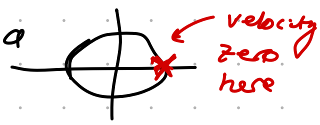
$$\frac{dW}{dz} = \frac{dW}{dw}\cdot\frac{dw}{dz} = \frac{w'(\zeta)}{z'(\zeta)}$$

Kutta condition: stagnation point in ζ at $\zeta = a \Rightarrow w'(a) = 0 \Rightarrow \Gamma = -4\pi U\sin\alpha$

Chapter 7 - Boundary Layers

Drag = ?

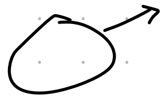
Word to Maths translation

word	maths	meaning
Inviscid	$\mu = 0$	Flow has no viscosity(μ) so can assume free slip conditions.
Steady	$\frac{\partial}{\partial t} = 0$	Flow does not vary in time
volumetric flow rate	$[Q] = \frac{m^3}{s}$ $Q = UA$ velocity, surface area	
vorticity	$\omega = \nabla \times u$	How much the fluid rotates / spins
conservation of mass	$\nabla \cdot u = 0$	Derived earlier
conservation of momentum	$\rho \frac{Du}{Dt} = -\nabla p + \mu \nabla^2 u - \rho g$	
unsteady incompressible potential flow	$\omega = 0$	$\phi(x,t)$ scalar f $u = \nabla\phi$, $\nabla \cdot u = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$ $\omega = \nabla \times u = \nabla \times \nabla \phi = 0 \rightarrow$ Bernoulli simplifies.
Torque	$T = \underbrace{2\pi r}_{\text{Area}} \cdot \underbrace{\sigma_{r\theta}}_{\text{Stress}} \cdot \underbrace{r}_{\text{moment arm}}$	Torque transmitted per unit axial length across any radius r
Integral form of conservation of momentum	$\int_{\Omega} \rho \frac{\partial u}{\partial t} d\Omega + \int_{\partial\Omega} \rho (u \cdot \hat{n}) u \, dA$	Just apply Reynold's transport thm w/ $b = \rho u$ [momentum], that is $\frac{D}{Dt}\int_{\Omega} b \, d\Omega = \int_{\Omega} \frac{\partial b}{\partial t} d\Omega + \int_{\partial\Omega} b (u \cdot \hat{n}) dA$
Kutta condition	circle $\longrightarrow$ plate $\zeta = a$ is stagnation point $\Rightarrow w'(\zeta) = 0$ (velocity)	Trailing edge is the rear stagnation point Basically, just set (complex) velocity $w' = 0$ at your boundary edge point $z = a$ 
velocity near the trailing edge	$\lim_{z \to a} \frac{dW}{dz}$ f use L'Hopital's rule	
cauchy stress tensor for inviscid fluid	on a circle $r = 1$ $T = -pI$	$F = \int_{\partial\Omega} T \cdot n \, ds = \int_0^{2\pi} -p\big\|_{r=1} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta =$ ______
viscosity	$[\mu] =$	μ = Newton seconds per square meter $[\mu] = \frac{Ns}{m^2} = \frac{Fs}{m^2} = \frac{MLT^{-2}T}{L^2} = \frac{M}{TL}$
Drag per unit length on the cylinder	$D = \int_{\partial\Omega} \hat{e}_x \cdot T \cdot \hat{n} \, dA$	where $T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Circulation

$C = x(s) = (R\cos(s), R\sin(s), 0)$

$\int_C u\cdot\hat{n}\,ds$ [points out]



$\hat{n} = r\hat{e}_r$

$\int_C u\cdot\hat{t}\,ds$ [points along]

$\hat{n} = r\hat{e}_\theta$

Questions

① Q sheet 7, Q 1c: $\nabla B = 0 \Rightarrow$ B constant in x, but you put $f(t)$

Problem Solving Method Log

SHEET 2

① 1. Read Section 2.8 in the lecture notes on mass conservation. In incompressible 3D flow, a pure straining motion is generated by the velocity field

$$\boldsymbol{u} = (\alpha x, \beta y, \gamma z)$$

(a) What are the components of the rate of strain tensor e_{ij}?

(b) What constraint does incompressibility place on α, β and γ?

(c) What is the vorticity of this flow?

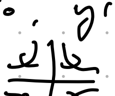
(d) Draw the streamlines of a two-dimensional incompressible flow ($\gamma = 0$) for $\alpha > 0$.

(e) Explain how you would expect this two-dimensional incompressible flow to deform an infinitesimal square fluid element located at the origin.

(a) $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} \alpha & & 0 \\ & \beta & \\ 0 & & \gamma \end{pmatrix}$

(b) $\nabla \cdot u = 0$

(c) $\omega = \nabla \times u$

(d) $\frac{\partial x}{\partial s} = u$, $\frac{\partial y}{\partial s} = v$, $x(0) = x_0$, $y(0) = y_0$, $\nabla \cdot u = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \frac{\beta}{\alpha} = -1$
and $u = \alpha x$ w/ $\alpha > 0$ $\Rightarrow$  direction of arrows specified.

② 2. Consider the motion of an infinitesimal rectangular fluid element of size δx by δy (see Figure 1) in a two-dimensional flow $\boldsymbol{u} = (u(x,y,t), v(x,y,t), 0)$. For simplicity, we will remove translation of the fluid element by considering flow relative to point A.

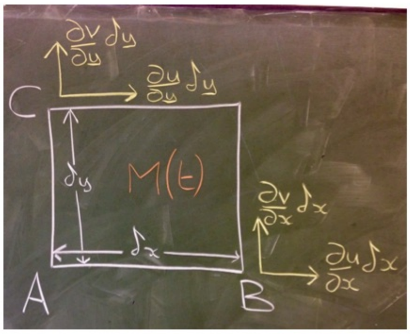


Figure 1: The velocity components relative to a point A in a material element $M(t)$ at a time t.

(a) Show that the extensional rate of strain $\frac{\mathrm{d}S_y}{\mathrm{d}t}$ due to increases in the legth of the side AC is given by e_{22}.

(b) Show that the shear components of the rate of strain act to drive the element from its rectangular shape by showing that the angle θ at CAB (see Figure 2) evolves according to

$$-\frac{\partial \theta}{\partial t} = 2e_{12}.$$

Hint: consider the vertical motion of the point B in Figure 2, which will increase the angle $\delta\beta$, and then the horizontal motion of the point C for $\delta\alpha$.

(a) Strain $= \frac{\text{extension}}{\text{original}} \Rightarrow \delta s_y = \frac{\delta y(t+\delta t) - \delta y(t)}{\delta y(t)}$

Taylor on (for AC) $\delta y(t+\delta t) = \delta y(t) + \frac{\partial v}{\partial y}\delta y\,\delta t$

$\left.\right\} \Rightarrow \frac{\partial s_y}{\partial t} = \frac{\partial v}{\partial y} = e_{22}$

$\because e_{22} = (\nabla u + \nabla u^T)_{22}$

(b)

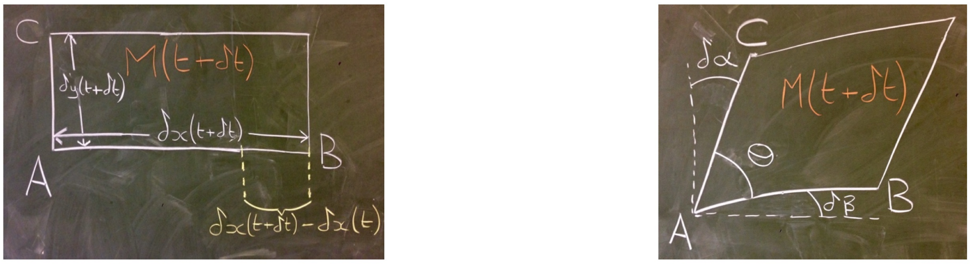
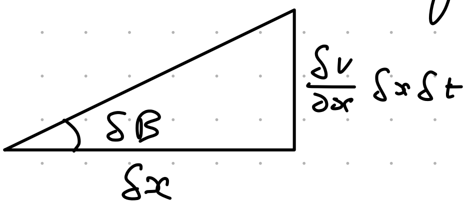


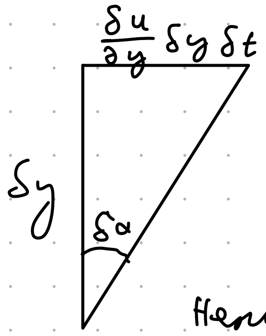
Figure 2: Influence of shear strain on the material element at a time $t+\delta t$, showing the deformation of an element from Figure 1 after time δt.

① Point B moves up by $\frac{\partial v}{\partial x}\delta x \delta t$ [distance = speed × time]



$\Rightarrow$ $\boxed{\delta B = \frac{\partial v}{\partial x}\delta t}$

② Point C moves right b



$\Rightarrow$ $\boxed{\delta\alpha = \frac{\partial u}{\partial y}\delta t}$

③ Angles sum to $\frac{\pi}{2}$

$$\delta\alpha + \theta + \delta B = \frac{\pi}{2}$$

so diff:

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \theta}{\partial t} + \frac{\partial B}{\partial t} = \frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial t} + \frac{\partial v}{\partial x}$$

Hence $-\frac{\partial\theta}{\partial t} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2e_{12}$

③ 3. (Adapted from Acheson p. 5) Consider a fluid in uniform rotation with angular velocity Ω so that in Cartesian coordinates the velocity is given by

$$u = -\Omega y, \quad v = \Omega x, \quad w = 0.$$

(a) Find the particle paths.

(b) Find the streamlines. Do they agree with the particle paths? Why?

(c) Calculate the acceleration a_i and interpret the result.

(a) $\underbrace{x \text{ Eulerian}}_{\text{global}}$, $\underbrace{X \text{ Lagrangian}}_{\text{local}}$, translate w/ $\left.\frac{\partial F}{\partial t}\right|_{\text{fixed } x} = u(F(x,t),t)$

So solve $\frac{\partial F_1}{\partial t} = -\Omega F_2$, $\frac{\partial F_2}{\partial t} = \Omega F_1$, $F_1(0) = X_1$, $F_2(0) = X_2$

$\rightsquigarrow$ $F_1(t) = X_1\cos(\Omega t) - X_2\sin(\Omega t)$
$F_2(t) = X_2\cos(\Omega t) + X_1\sin(\Omega t)$

Solve w/ sub & guess $e^{\lambda t}$
$\lambda = \pm i\Omega$

$F_1^2 + F_2^2 = X_1^2 + X_2^2$ $\Rightarrow$ streamlines circle radius $(X_1^2 + X_2^2)^{\frac{1}{2}}$

(b) $\frac{dX}{ds} = -\Omega Y$, $\frac{\partial Y}{\partial s} = \Omega X$ $\rightsquigarrow$ Same!

(c) TWO METHODS:

Uniform rotation $\Rightarrow \frac{\partial u}{\partial t} = 0$

(i) Material Derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + (u\cdot\nabla)$

$a_1 = \ldots$, $a_2 = \ldots$

(ii) Lagrangian: $\frac{\partial^2 F_1}{\partial t^2} = \dots$ $\qquad \frac{\partial^2 F_2}{\partial t^2} = \dots$

acceleration centripetal (points inwards) w/ magnitude $\omega^2 r$

$\boxed{\text{SHEET 3}}$

4 1. Consider the conservation law for a dissolved species, like salt in water or carbon dioxide in air. For now, we will assume that there are no chemical reactions in the fluid to generate the species.

(a) Identify the quantities needed to track the concentration of such a species in the presence of a flow. What form does the surface rate-of-exchange term take?

(b) Write the law of conservation of the species inside any volume Ω in integral form. Present an interpretation of each term in your expression.

(c) Apply the divergence theorem to any terms in the integral representation of the conservation law in the previous section.

(d) Write the conservation law in a differential form. Present an interpretation of each term in your expression.

(e) Search about Fick's law online and read about it. How do you think Fick's law could be applicable here?

(a) c = mass of species per unit volume.
volumetric generation rate zero $\because$ no reactions in fluid
q = surface source term is 1 tensor rank above c [scalar] $\Rightarrow$ vector.

(b) Use volume/surface breakdown & Reynold's transport thm

$$\frac{D}{Dt}\int_\Omega c\,d\Omega = \underbrace{\cancel{\int_\Omega q^c\,d\Omega}}_{\text{volume sources}} + \underbrace{\int_{\partial\Omega} q\cdot\hat{n}\,dA}_{\text{surface sources}} = \underbrace{\int_\Omega \frac{\partial c}{\partial t}\,d\Omega + \int_{\partial\Omega} c(u\cdot\hat{n})\,dA}_{\text{reynold's transport thm}}$$

(c) $\displaystyle\int_{\partial\Omega} c(u\cdot\hat{n})\,dA = \int_\Omega \nabla\cdot(cu)\,d\Omega \qquad \int_\Omega \nabla\cdot q\,dV$

(d) so $\nabla\cdot q = \dfrac{\partial c}{\partial t} + \nabla\cdot(cu)$

(e) $q = D\nabla c$ where D = diffusivity.

5 2. Consider two chemical species A and B that undergo a simple reaction

$$A \rightleftharpoons B.$$

The chemical kinetics for this reaction in an homogeneous mixture, and in the absence of flow, may be written as

$$\frac{\mathrm{d}[A]}{\mathrm{d}t} = -\frac{\mathrm{d}[B]}{\mathrm{d}t} = -k_A[A] + k_B[B],$$

where $[A]$ and $[B]$ are the molar quantities of A and B per unit volume, respectively, and k_A and k_B are the reaction rate constants.

(a) Write the conservation law in integral form for species A and B inside an arbitrary volume Ω. Define any quantities you need to develop this expression.

(b) Convert the conseevation law to differential form.

(a) Do each species separately & use same source/volume & reynold's breakd.

$$\int_\Omega \frac{\partial c}{\partial t}\,d\Omega + \int_{\partial\Omega} c(u\cdot\hat{n})\,dA = \underbrace{\int_\Omega q^c\,d\Omega}_{\text{volume}} + \underbrace{\int_{\partial\Omega} h^c\cdot\hat{n}\,dA}_{\text{surface}}$$

For A: $\int_{\Omega} \frac{\partial [A]}{\partial t} d\Omega + \int_{\partial\Omega} [A](u\cdot\hat{n})\, dA = \int_{\Omega} -k_A[A] + k_B[B]\, d\Omega + \int_{\partial\Omega} h^A\cdot\hat{n}\, dA$

same for B, then div term & write as div

(b) write as $\frac{\partial [A]}{\partial t} + \nabla\cdot([A]u) = -k_A[A] + k_B[B] + \nabla\cdot h^A$, same 4 B

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3. Here we will develop the consequence of conservation of angular momentum (this is an alternative derivation that will lead to the same results derived in Section 2.10 of the lecture notes).

   (a) Using the $\boldsymbol{x}\times\rho\boldsymbol{u}$ as the density of angular momentum, $\boldsymbol{x}\times\rho\boldsymbol{g}$ as the volumetric source term and $\boldsymbol{x}\times(\boldsymbol{T}\cdot\hat{\boldsymbol{n}})$ as the surface source term ($\hat{\boldsymbol{n}}$ is the unit outward normal to the volume surface), write the conservation law for angular momentum in integral form.

   (b) Use the divergence theorem to convert it to differential form.

   (c) Simplify the differential form by using (i) the product rule for differentiation, (ii) the law of conservation of linear momentum, and (iii) the anti-symmetry of the alternating tensor, to reduce the differential form to $\epsilon_{ijk}T_{jk} = 0$.

   (d) Deduce from the anti-symmetry of ϵ_{ijk} in the previous section that the stress tensor T_{jk} must be symmetric.

(a) Apply result of cauchy tetrahedron argument: $\frac{\partial b}{\partial t} + \nabla\cdot(bu) = q^B + \nabla\cdot T^B$

In integral form $\int_{\Omega}\frac{\partial b}{\partial t}\, d\Omega + \int_{\partial\Omega} b(u\cdot\hat{n})\, dA = \int_{\Omega} q^B\, d\Omega + \int_{\partial\Omega} \underbrace{h^B(\hat{n})}_{T^B_{\dots j}n_j}\, dA$

So w/ div thm

$$\int_{\Omega}\left(\frac{\partial b}{\partial t} + \nabla\cdot(bu)\right) d\Omega = \int_{\Omega} q^B + \frac{\partial T^B_{\dots j}}{\partial x_j}\, d\Omega$$

Here $b = x\times\rho u$, $q^B = x\times\rho g$, $h^B = x\times(T\cdot\hat{n})$

Index: $b_i = \varepsilon_{ijk}x_j\rho u_k$, $q^B_i = \rho\,\varepsilon_{ijk}x_j g_k$, $T^B_{im} = \varepsilon_{ijk}x_j T_{km}$

(b) $$\underbrace{\frac{\partial(\varepsilon_{ijk}x_j\rho u_k)}{\partial t}}_{1} + \underbrace{\frac{\partial(\varepsilon_{ijk}x_j\rho u_k u_m)}{\partial x_m}}_{2} = \rho\,\varepsilon_{ijk}x_j g_k + \underbrace{\frac{\partial(\varepsilon_{ijk}x_j T_{km})}{\partial x_m}}_{3}$$

(c i)
$$\overbrace{\rho\,\varepsilon_{ijk}u_j u_k + \underbrace{\rho\,\varepsilon_{ijk}x_j\frac{\partial u_k}{\partial t}}_{a}}^{1} + \overbrace{\rho\,\varepsilon_{ijk}\frac{\partial x_j}{\partial x_m}u_k u_m + \underbrace{\rho\,\varepsilon_{ijk}x_j\frac{\partial(u_k u_m)}{\partial x_m}}_{b}}^{2}$$
$$= \underbrace{\rho\,\varepsilon_{ijk}x_j g_k}_{c} + \underbrace{\varepsilon_{ijk}\frac{\partial x_j}{\partial x_m}T_{km} + \overbrace{\varepsilon_{ijk}x_j\frac{\partial T_{km}}{\partial x_m}}^{d}}_{3}$$

(ii)
$$\varepsilon_{ijk}x_j\underbrace{\left(\overbrace{\rho\frac{\partial u_k}{\partial t}}^{a} + \overbrace{\rho\frac{\partial(u_k u_m)}{\partial x_m}}^{b} - \overbrace{\rho g_k}^{c} - \overbrace{\frac{\partial T_{km}}{\partial x_m}}^{d}\right)}_{=0\ \text{(momentum cons.)}} + \rho\,\varepsilon_{ijk}u_j u_k + \underbrace{\overbrace{\frac{\partial x_j}{\partial x_m}}^{\delta_{jm}}}_{j=m\ \text{only}}\left(\rho\,\varepsilon_{ijk}u_k u_m - \varepsilon_{ijk}T_{km}\right) = 0$$

(iii)
$$\overbrace{\rho\,\varepsilon_{ijk}u_j u_k + \underbrace{\rho\,\varepsilon_{ijk}u_k u_j}_{-\varepsilon_{ikj}u_k u_j = -\varepsilon_{ijk}u_j u_k}}^{=0} - \varepsilon_{ijk}T_{kj} = 0 \Rightarrow \varepsilon_{ijk}T_{kj} = 0$$

(d) Evaluating

$$\left.\begin{array}{ll} i=1 \Rightarrow & T_{23} - T_{32} = 0 \\ i=2 \Rightarrow & T_{31} - T_{13} = 0 \\ i=3 \Rightarrow & T_{12} - T_{21} = 0 \end{array}\right\} \quad T_{ij} = T_{ji} \quad \forall ij \Rightarrow T \text{ symmetric}$$

Note: This expression is calculated componentwise in i, leave it as the free index. Then $h^B\cdot\hat{n}$ term is sorted out by the cauchy tetrahedron argument.

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(a) Show that the flow given by (2) is incompressible.

(b) Find the components of the velocity on the surface of the sphere.

(c) Without substituting the velocity or pressure in (1), write an expression for the force on the sphere in terms of an integral in θ. Keep this integral in terms of the components of stress.

(d) Evaluate the stress components needed to determine the force on the surface of the sphere.

(e) Evaluate the integral in θ to find the force.

equations defined in Ex Sheet 3 Q4

in spherical coordinates (r, θ, ϕ)

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

(a) WTS $\nabla\cdot u = 0$, but in axisymmetric spherical coordinates, told that

$$\nabla\cdot u = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta}$$

$$u_r = \frac{U\cos\theta}{2r^2}\left(\frac{a^3}{r} - 3ar + 2r^2\right)$$

$$u_\theta = -\frac{U\sin\theta}{4r}\left(-\frac{a^3}{r^2} - 3a + 4r\right)$$

So sub

$$\nabla\cdot u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{2}U\cos\theta\left(\frac{a^3}{r} - 3ar + 2r^2\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-U\sin^2\theta}{4r}\left(-\frac{a^3}{r^2} - 3a + 4r\right)\right)$$

$$= \frac{U\cos\theta}{2r^2}\left(-\frac{a^3}{r^2} - 3a + 4r\right) + \frac{U}{4r^2\cancel{\sin\theta}}\left(\frac{a^3}{r^2} + 3a - 4r\right)\cdot 2\cancel{\sin\theta}\cos\theta$$

$$= 0$$

(b) set $r = a$, $u_r = \frac{U\cos\theta}{2a^2}\left(a^2 - 3a^2 + 2a^2\right) = 0$

$$u_\theta = -\frac{U\sin\theta}{4a}\left(-a - 3a + 4a\right) = 0 \Rightarrow u = 0 \text{ on } r = a$$

stress tensor

(c) $F = \int_S \sigma\cdot\hat{n}\, dA$, $\sigma = \sigma_{rr}\hat{e}_r\hat{e}_r + \sigma_{\theta r}(\hat{e}_r\hat{e}_\theta + \hat{e}_\theta\hat{e}_r) + \sigma_{\theta\theta}\hat{e}_\theta\hat{e}_\theta + \sigma_{\phi\phi}\hat{e}_\phi\hat{e}_\phi$

note: $\sigma\cdot\underbrace{\hat{n}}_{\hat{e}_r} = \begin{pmatrix}\sigma_{rr} & \sigma_{r\theta} & \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \\ & & \sigma_{\phi\phi}\end{pmatrix}\cdot\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{=\hat{e}_r} = \begin{pmatrix}\sigma_{rr}\\ \sigma_{\theta r}\\ 0\end{pmatrix}$ so $\sigma\cdot\hat{n} = \sigma_{rr}\hat{e}_r + \sigma_{\theta r}\hat{e}_\theta$

$$\therefore \underbrace{F}_{\text{vector}} = \int_S \sigma\cdot\hat{n}\, dA = \int_0^{2\pi}\int_0^{\pi}(\sigma_{rr}\hat{e}_r + \sigma_{\theta r}\hat{e}_\theta)\, r^2\sin\theta\, d\theta\, d\phi$$

really 2 integrals one in $\hat{e}_r$, one in $\hat{e}_\theta$

$$= 2\pi a^2\int_0^{\pi}(\sigma_{rr}\hat{e}_r + \sigma_{\theta r}\hat{e}_\theta)\sin\theta\, d\theta \qquad [r = a \because \text{surface int.}]$$

(d) $\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} = \frac{3}{2}\rho V a\frac{\cos\theta}{a^2} + 2\mu\frac{\partial}{\partial r}\left(\frac{U\cos\theta}{2r^2}\left(\frac{a^3}{r} - 3ar + 2r^2\right)\right)$

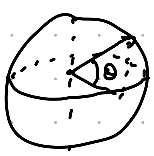
$= \dots$

$= \frac{3\mu U\cos\theta}{2a}$ at $r = a$

$$\sigma_{\theta r} = \mu r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial u_r}{\partial\theta} = \dots = -\frac{3\mu U\sin\theta}{2a}$$

(e) $\hat{e}_r = \hat{z}\cos\theta + \hat{x}\sin\theta$

$\hat{e}_\theta = -\hat{z}\sin\theta + \hat{x}\cos\theta$



$$\therefore F = 2\pi a^2\cdot\frac{3\mu U}{2a}\int_0^{\pi}\left[\cos\theta(\hat{z}\cos\theta + \hat{x}\sin\theta) - \sin\theta(-\hat{z}\sin\theta + \hat{x}\cos\theta)\right]\sin\theta\, d\theta$$

$$= 6\pi\mu a U\hat{z}$$

(8) $\boxed{\text{SHEET 4}}$

1. Consider the steady two-dimensional flow in cylindrical polar coordinates given by

$$u = U\cos\theta, \quad v = -U\sin\theta + \frac{\Gamma}{2\pi r}, \quad w = 0,$$

where U and Γ are constants. Ignore any body (or volumetric) forces.

(a) Show that this flow is incompressible.

(b) Show that it is possible to determine th epressure field so that the flow satisfies conservation of momentum everywhere (except, perhaps, at $r = 0$).

(a) $\nabla \cdot u$ in cylyndrical

(b) viscous terms $= 0$, steady state $\Rightarrow \frac{\partial}{\partial t} = 0$ ... $p =$ ———

(9)

2. The flow in Question 1 is generated by the action of a point force at the origin on an otherwise uniform flow $\boldsymbol{u} = U\hat{\boldsymbol{u}}_x$.

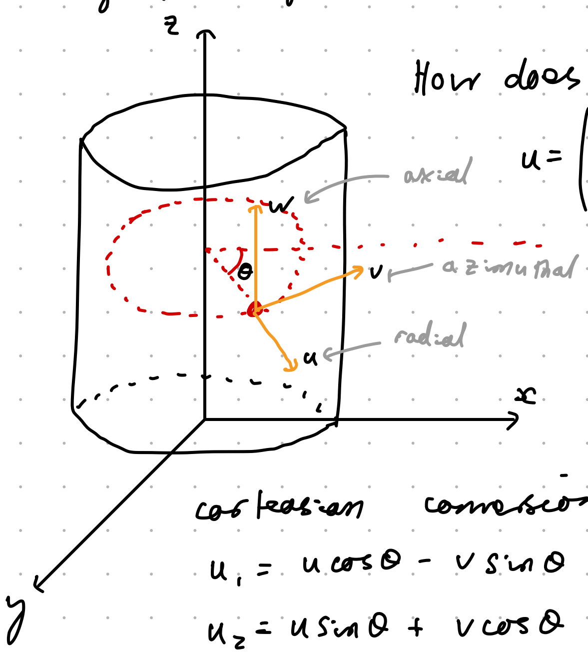
(a) To see this show that the Cartesian components of the flow in Question 1 are

$$u = U - \frac{\Gamma}{2\pi r}\sin\theta, \quad v = \frac{\Gamma}{2\pi r}\cos\theta, \quad w = 0.$$

The force disappears when $\Gamma = 0$, in which case $\boldsymbol{u} = U\hat{\boldsymbol{u}}_x$ everywhere.

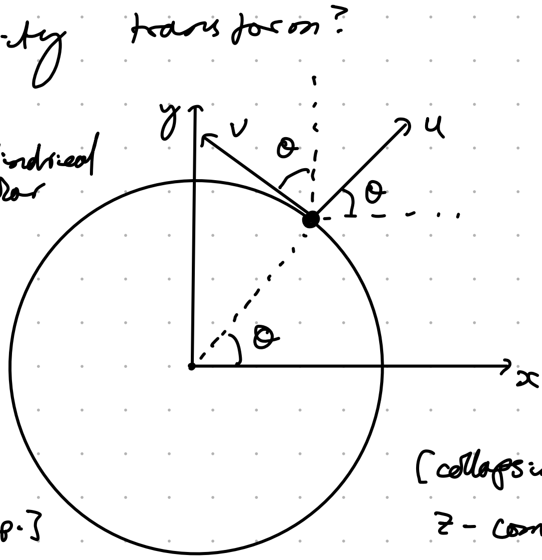
(b) Consider the Eulerian volume consisting of a cylinder of radius R and infinite length so that a two-dimensional treatment is possible. Use the integral form of momentum balance to determine the force acting at the origin. Assume that the viscous stresses do not influence the momentum balance (which we state without proof to be exactly true).

(a) In cylindrical polar coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$



How does velocity transform?

$u = \begin{pmatrix} u \\ w \\ z \end{pmatrix}$ in cylindrical polar



cartesian conversion

$u_1 = u\cos\theta - v\sin\theta$ [x comp.]

$u_2 = u\sin\theta + v\cos\theta$ [y comp.]

(collapsing the z-component)

so here, $u_1 = U\cos^2\theta + U\sin^2\theta - \frac{\Gamma\sin\theta}{2\pi r}$, $u_2 = U\sin\theta\cos\theta - U\sin\theta\cos\theta + \frac{\Gamma\cos\theta}{2\pi r}$

(b) use dA formulation $\because$ you have normal

$$\int_{\Omega} \frac{\partial(\rho u_i)}{\partial t}\, d\Omega + \int_{\partial\Omega} \rho u_i (u\cdot\hat{n})\, dA = \int_{\partial\Omega} -p\hat{n}\, dA + F_i$$

$\Rightarrow F_1 = \ ?$ & sub w/ $\hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Rightarrow u\cdot\hat{n} = \cos\theta$

and polar so $dA \to r\,dr\,d\theta$

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4. Consider a viscous fluid of density ρ and viscosity μ between two infinite coaxial cylinders of radius R_i and R_o, respectively, as shown in Figure 1. The inner cylinder spins with angular velocity ω_i, while the outer one spins with angular velocity ω_o. We can assume that the fluid has reached a steady state.

   (a) Which component(s) of velocity in cylindrical polar coordinates do you expect to be non-zero? Which coordinates would it (they) depend on?

   (b) Simplify the Navier-Stokes equations in cylindrical coordinates to the case considered in the previous section and derive differential equations for the non-zero component(s) of the velocity.

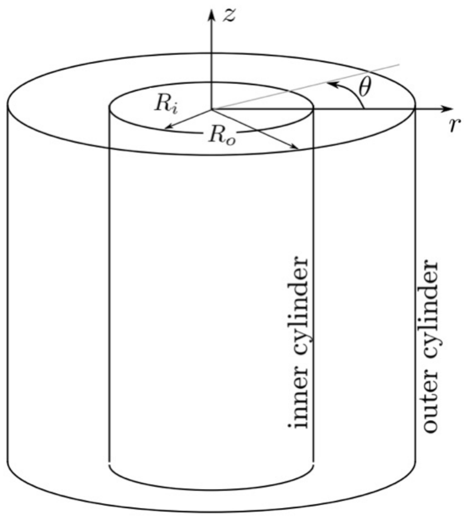

Figure 1: Influence of shear strain on the material element at a time $t + \delta t$, showing the deformation of an element from Figure 1 after time δt.

   (c) What would be the boundary conditions on the profile(s) of the non-zero component(s) of the velocity?

   (d) Solve the differential equations determined in 4(b) and use the boundary conditions from 4(c) to derive the velocity profile.

(a) only azimuthal velocity component that depends on r only. $\Rightarrow u = v(r)\,\hat{e}_\theta$ i.e. $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ v(r) \\ 0 \end{pmatrix}$

(b) Assumptions yield $-\frac{\rho v^2}{r} = -\frac{\partial p}{\partial r}$, $\underbrace{\frac{\partial p}{\partial \theta}}_{=0} = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(vr)\right)$

(c) $v(r = R_i) = \omega_i R_i$, $v(r = R_o) = \omega_o R_o$

(d) $\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) = 0 \Rightarrow \frac{\partial}{\partial r}(rv) = Ar \Rightarrow v = \frac{1}{2}Ar + \frac{B}{r}$

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5. Consider a semi-infinite layer of fluid in the region $y \geq 0$ in two-dimensional Cartesian coordinates. The oscillations of a solid plate on the x-axis $y = 0$ drives a flow in the fluid. Consider this plate to move along the x direction with speed

$$U(t) = A\cos\omega t,$$

where A and ω are constants. The profile of fluid velocity in this case is of the form

$$\boldsymbol{u} = u(y,t)\hat{\mathbf{e}}_x.$$

   (a) Simplify the Navier-Stokes equations in Cartesian coordinates for the unsteady one-dimensional flow $\boldsymbol{u}$.

   (b) A solution may be found by considering that the flow velocity also oscillates sinusoidally with the same angular frequency ω, i.e.

$$u(y,t) = \mathrm{Re}\left(f(y)\mathrm{e}^{i\omega t}\right), \tag{1}$$

   where the complex function $f(y)$ is to be determined. Substitute (1) in the simplified Navier-Stokes equations to derive an ordinary differential equation for f.

   (c) Solve the ordinary differential equation for f. You may need to use $i = \mathrm{e}^{i\frac{\pi}{2}}$ and $\sqrt{i} = \pm\mathrm{e}^{i\frac{\pi}{4}}$.

   (d) To determine the constants of integration apply the no-slip condition at $y = 0$ and that the fluid is stagnant at $y = \infty$. Write the final velocity profile.

   (e) Calculate the tangential (i.e. x) component of the traction force exerted by the fluid on the oscillating wall. What is the phase difference between the force experienced by the wall and its velocity?

(a) yields $\rho \frac{\partial u}{\partial t} = -\nabla p + \mu \nabla^2 u$ $u = u(y,t)\hat{e}_x$

$\nabla p = \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{pmatrix}$, $\frac{\partial p}{\partial y} = 0$ $\because$ no y flow $\Rightarrow$ $p(x)$ but no external pressure gradient, so $p = $ const, say zero. $\frac{\partial p}{\partial x} = 0$

$\nabla^2 u = \left(\cancel{\frac{\partial^2}{\partial x^2}} + \frac{\partial^2}{\partial y^2} \right) u \Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$\nu = \frac{\mu}{\rho}$

initial: $u(y=0, t) = A\cos(\omega t)$,
$u(y \to \infty, t) = 0$

(b) Sub full complex expression, take real part at end.

$\frac{\partial^2 f}{\partial y^2} = \frac{i\omega}{\nu} f \iff \lambda^2 = \frac{i\omega}{\nu} \iff \lambda = \pm \sqrt{\frac{\omega}{\nu}}\, e^{\frac{i\pi}{4}} \leftarrow \sqrt{i}$

Therefore $f(y) = b_1 e^{\lambda y} + b_2 e^{-\lambda y}$

(d) $f(y=0, 0) = A \Rightarrow b_1 + b_2 = A$, $f(y \to \infty, t) \Rightarrow b_1 = 0$

so $f(y) = Ae^{-\lambda y} \Rightarrow u(y,t) = \mathrm{Re}(Ae^{-\lambda y} e^{i\omega t})$

(e) Traction force along x: $\mu \frac{\partial u}{\partial y}$ $z \in \mathbb{C} \Rightarrow \mathrm{Re}(z) = \frac{z + \bar{z}}{2}$

so $u(y,t) = \frac{A}{2}(e^{-\lambda y + i\omega t} + e^{-\bar{\lambda} y - i\omega t})$ $\lambda + \bar{\lambda} =$

$$\text{Traction} = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\mu A}{2}(-\lambda e^{i\omega t} - \bar{\lambda} e^{-i\omega t})$$
$$= -\frac{\mu A}{2}\sqrt{\frac{\omega}{\nu}}\left(e^{i\omega t + \frac{\pi}{4}}\right)$$
$$= -\frac{\mu A}{2}\sqrt{\frac{\omega}{\nu}}\cos\left(\omega t + \frac{\pi}{4}\right) \Rightarrow \text{phase diff. } \frac{5\pi}{4}$$

---

(12) $\boxed{\text{SHEET 5}}$

3. Consider Stokes second problem, where a semi-infinite layer of fluid in the region $y \geq 0$ moves due to oscillations of a solid plate at $y = 0$. Consider this plate to move along the x direction with speed

$$U(t) = A\cos\omega t,$$

where A and ω are constants. The profile of fluid velocity in this case is of the form

$$\boldsymbol{u} = u(y,t)\hat{\boldsymbol{e}}_x,$$

where $u(y,t)$ satisfies (1a) with boundary conditions

$$u(0,t) = U(t), \qquad t > 0, \qquad (2a)$$
$$u(y \to \infty, t) = 0, \qquad t > 0. \qquad (2b)$$

$m = \frac{N}{ms^{-2}}$ $F = ma$ $\therefore [kg] =$

(a) Construct transformations to non-dimensionalise u, y and t based on the parameters of the problem A, ω and ν. $[A] = \frac{L}{T}$, $[\omega] = \frac{1}{T}$, $[\nu] = \left[\frac{\mu}{\rho}\right] = \frac{Ns/m^2}{kg/m^3} = \frac{Nsm}{N/ms^2} = \frac{L^2}{T}$

(b) Substitute these transformations into (1a) and (2) to obtain the dimensionless form of the problem. How many dimensionless parameters remain in the problem?

(c) How would the time-dependent drag on the plate depend on the parameters A, μ, ν and ω?

(a) $u = A\tilde{u}$, $t = \frac{1}{\omega}\tilde{t}$, $y = \sqrt{\frac{\nu}{\omega}}\,\tilde{y}$

two of these $\because$ diff wrt y twice

only need 1 of these $\because$ cancel once.

careful w/ derivatives $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial \tilde{u}}{\partial \tilde{t}} \cdot \frac{\partial u}{\partial \tilde{u}} \cdot \frac{\partial \tilde{t}}{\partial t} = \nu \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \cdot \left(\frac{\partial \tilde{y}}{\partial y}\right)^2 \frac{\partial u}{\partial \tilde{u}}$

(c) Drag per unit area $= D = \overset{\text{viscosity / resistance}}{\mu} \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \left.\frac{\partial \tilde{u}}{\partial \tilde{y}} \cdot \frac{\partial u}{\partial \tilde{u}} \cdot \frac{\partial \tilde{y}}{\partial y}\right|_{y=0}$

$= \frac{\mu A}{\sqrt{\nu/\omega}} \left.\frac{\partial \tilde{u}}{\partial \tilde{y}}\right|_{y=0}$ parameter independent so dimensionless constant

$\Rightarrow D = \mu A \sqrt{\frac{\omega}{\nu}} \times$ const.

SHEET 6

4. Consider a sphere with a given pulsating radius $R(t)$ (e.g. a bubble), which drives a pulsating spherically symmetric flow in the surrounding fluid $\boldsymbol{u} = u(r,t)\hat{\boldsymbol{u}}_r$.

   (a) Use conservation of mass and a suitable boundary condition on the surface of the sphere to determine an expression for $u(r,t)$.

   (b) Could we use the Bernoulli equation to determine the pressure in the fluid? Explain why. If yes find the pressure, you may neglect volumetric forces and assume that the pressure far away from the sphere is p_∞.

   (c) The radial component of the momentum balance for $u(r,t)$ reads

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \frac{2u}{r^2}\right).$$

   Determine the pressure using this equation. If you could find the pressure in b), does it agree with the pressure you found here?

   (d) Does the expression for the pressure depend on the fluid viscosity? If not, why?

(a) $\nabla \cdot u = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,t)\right) = 0 \Rightarrow u(r,t) = \frac{C(t)}{r^2}$

B.C. $u(R(t), t) = R'(t) \Rightarrow R' = \frac{C(t)}{R^2} \Rightarrow u(r,t) = \dot{R}\frac{R^2}{r^2}$

(b) here $\omega = \nabla \times u = 0 \Rightarrow$ USE POTENTIAL FLOW !!!

$u = \nabla\phi \Rightarrow \nabla\cdot u = \nabla\cdot\nabla\phi = \nabla^2\phi = 0$ so RHS bernoulli $= 0$

$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r = u = \frac{\partial\phi}{\partial r} = \dot{R}\frac{R^2}{r^2} \Rightarrow \phi(r,t) = -\dot{R}\frac{R^2}{r}$

Bernoulli $\Rightarrow \nabla B = 0 \Rightarrow \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p =$ const. in r

limit $r \to \infty \Rightarrow p \to p_\infty \Rightarrow$ const $= p_\infty$, sub.