Complex Analysis Summary

Chapter 1 – Review of Basic complex Analysis I

(Take $h = \alpha$, $h = i\alpha$ $\longrightarrow$ CR) exists

① **Def 1.1:** $\Omega \subset \mathbb{C}$ open, $f : \Omega \longrightarrow \mathbb{C}$ complex diff at $z \in \Omega$ if $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$

Note: View $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$: (f complex diff) $\Longleftrightarrow$ (f real diff (MVC) & cauchy riemann)

- $f_{\bar z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right)$; $f_z := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right) \Rightarrow$ CR: $f_{\bar z} = 0$, $f'(z) = f_z$
- $f : \Omega \longrightarrow \mathbb{C}$ holomorphic if f complex diff $\forall z \in \Omega$. $\Omega = \mathbb{C} \Rightarrow f$ entire

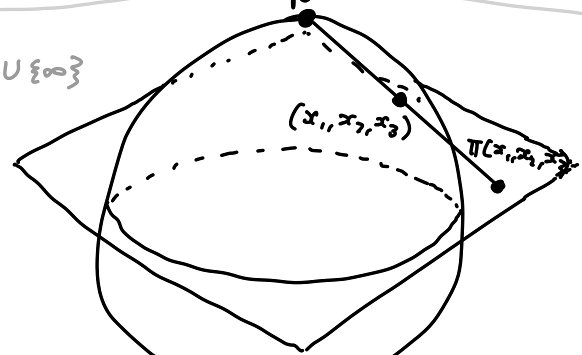
Chapter 2 – Möbius Transformations

② **Def 2.1:** Stereographic projection $\pi : S^2 \longrightarrow \mathbb{C}_\infty$ ($= \mathbb{C} \cup \{\infty\}$)

$$\pi(x_1, x_2, x_3) = \frac{x_1 + i x_2}{1 - x_3}$$

bijection! ($\pi(N) = \infty$)

$$\pi^{-1}(x + iy) = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1 + |z|^2}\right)$$



③ **Def 2.4:** Möbius transformations $f : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ are $f(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$

Note: bijective, cts & cts inverse $\Rightarrow$ homeomorphisms $\Rightarrow$ {möbius transformations} group under composition

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f_M(z) = \frac{az+b}{cz+d}$$ group isomorphism btwn $\wr$ & $PSL(2, \mathbb{C})$

set det $= \pm 1$ ← divide out by kernel of map

$$PSL(2, \mathbb{C}) := SL(2, \mathbb{C}) / \{\pm I\}$$

$\boxed{GL(2, \mathbb{C}) \supset SL(2, \mathbb{C})}$

Thm 2.10: circles in $\mathbb{C}_\infty$ preserved under möbius transformations

Pf: obvs (i)–(iii), (iv) $\Rightarrow$ its a $180°$ rotation & circles/lines preserved by stereographic projection

④ **Def 2.8:** Four elementary möbius transformations

(i) Translations $z \longmapsto z + b$, $b \in \mathbb{C}$

(ii) Rotations $z \longmapsto e^{i\theta} z$, $\theta \in \mathbb{R}$

(iii) Dilations $z \longmapsto \lambda z$, $\lambda > 1$ expansion, $0 < \lambda < 1$ contraction

(iv) Complex Inversion $z \longmapsto \frac{1}{z}$, $180°$ in x, axis

every möbius transformation can be written as the composition of these elementary transf.

⑤ **Lem 2.12:** Every möbius transformation (bar Id) has $\in \{1, 2\}$ fixed pts. If z_1, z_2, z_3 distinct w/ $f(z_i) = z_i \Rightarrow f(z) = z$ (identity)

Pf: $f(z) = \frac{az+b}{cz+d}$. $a = d \neq 0$, $b = c = 0 \Rightarrow$ identity, assume not these cases:

① $c = 0 \Rightarrow f(z) = \frac{a}{d} z + \frac{b}{d}$. Fixed pts: $z = \infty$, $z = \frac{b}{d - a}$ if $a \neq d$

② $c \neq 0 \Rightarrow$ set $= z$, quadratic formula $\Rightarrow$ two solutions which could coincide ■

⑥ **Lem 2.13:** $\exists$ möbius T mapping $z_1, z_2, z_3 \in \mathbb{C}_\infty$ distinct to $1, 0, \infty$ respectively.

$z_1 \longmapsto 1$, $z_2 \longmapsto 0$, $z_3 \longmapsto \infty$

$$f(z) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

Pf: construct!

⑦ **Thm 2.11:** $i \in \{1, 2, 3\}$, z_i distinct, w_i distinct in $\mathbb{C}_\infty$, $\exists$ unique möbius f s.t. $f(z_i) = w_i$

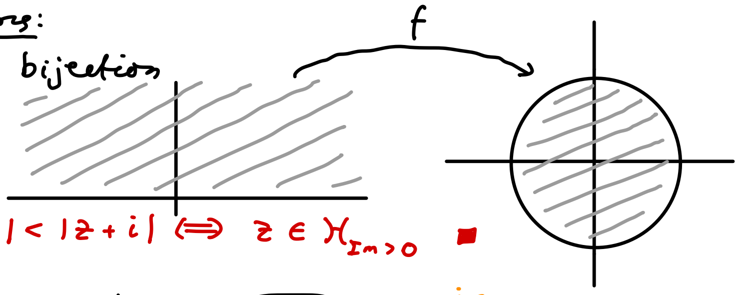
Pf: Existence: $f_1(z_1, z_2, z_3) = (1, 0, \infty)$, $f_2(w_1, w_2, w_3) = (1, 0, \infty)$. $f_2^{-1} \circ f_1$ works

Uniqueness: f, g both $z_i \longmapsto w_i$. $g^{-1} \circ f$ has 3 fixed points $\Rightarrow g^{-1} \circ f$ identity ■

⑧ Special classes of Möbius transformations:

- Cayley transform: $\mathcal{H}_{Im>0} \longrightarrow D$ bijection

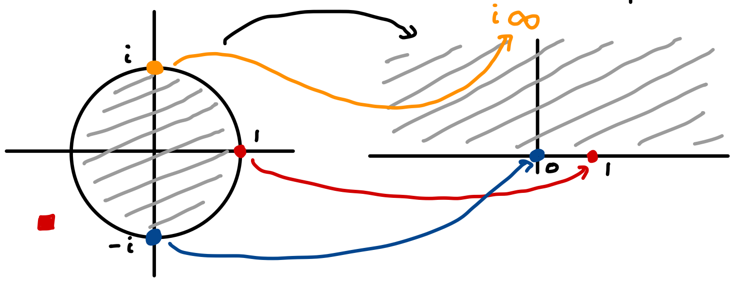
$$f(z) = \frac{z-i}{z+i}$$



Pf: $f(z) \in D \Leftrightarrow |f(z)| < 1 \Leftrightarrow |z-i| < |z+i| \Leftrightarrow z \in \mathcal{H}_{Im>0}$ ■

- Bare hands: $D \longrightarrow \mathcal{H}_{Im>0}$

$$f(z) = \frac{z+i}{iz+1}$$



Pf: use ⑥ $1 \longmapsto 1,\ -i \longmapsto 0,\ i \longmapsto \infty$ ■

- Bijections: $D \longrightarrow D$, $w \in \mathbb{C}$, $|w| < 1$

$$f(z) = \frac{z-w}{\overline{w}z-1}(e^{i\theta})$$

Schwarz lemma $\Rightarrow$ These are the ONLY $D \to D$

extreme rigidity of complex analysis.

Pf: $|z-w|^2 = |\overline{w}z-1|^2 - \underbrace{(1-|z|^2)(1-|w|^2)}_{>0}$

$$|f(z)|^2 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|\overline{w}z-1|^2}$$

$1-|w|^2>0 \Rightarrow \begin{cases} |f(z)| < 1 \Leftrightarrow |z| < 1 \\ f(z) = 1 \Leftrightarrow |z| = 1 \end{cases}$ ■

- Bijections: $\mathcal{H}_{Im>0} \longrightarrow \mathcal{H}_{Im>0}$

$$PSL(2,\mathbb{R}) \cong \left\{ \begin{array}{c} \text{bijections from} \\ \mathcal{H}_{Im>0} \longmapsto \mathcal{H}_{Im>0} \end{array} \right\}$$

$\cap$

$PSL(2,\mathbb{C})$

Pf: Note $a, b, c, d \in \mathbb{R}$ so

$$f(z) = \frac{(az+b)(c\overline{z}+d)}{(cz+d)(c\overline{z}+d)} = \frac{\overbrace{ac|z|^2}^{\text{Real}} + adz + bc\overline{z} + \overbrace{bd}^{\text{real}}}{|cz+d|^2}$$

$$Im(f(z)) = \frac{\overbrace{ad-bc}^{SL(2,\mathbb{R}) \Rightarrow \det = 1}}{|cz+d|^2} Im(z) \Rightarrow \left(\begin{array}{c} Im(z) > 0 \\ \Leftrightarrow \\ Im(f(z)) > 0 \end{array} \right)$$

$\Downarrow$

$(z \in \mathcal{H}_{Im>0} \Leftrightarrow f(z) \in \mathcal{H}_{Im>0})$ ■

$\boxed{\text{Domain} \overset{?}{=} \left(\begin{array}{c} \text{non-empty} \\ \text{open \& connected} \\ \text{subset} \end{array} \right) \text{ (Based on context)}}$

⑨ Def 2.25: $f: \Omega \longrightarrow \mathbb{C}$ conformal map if f holo. & $f'(z) \neq 0$

open; preserves angles; $\forall z \in \Omega$

↳ $z \longmapsto z^2$ not injective so define...

⑩ Def 2.26: $f: \Omega_1 \longrightarrow \Omega_2$, $\Omega_1, \Omega_2 \subset \mathbb{C}$ open is biholomorphic if f is a bijection s.t. both f and f^{-1} are conformal maps.

⑪ Def 2.27: Two domains Ω_1, Ω_2 are conformally equivalent if $\exists$ biholo $\varphi: \Omega_1 \longrightarrow \Omega_2$.

Note: ?? $\Rightarrow$ every bijective holomorphic f is automatically biholomorphic

↳ f^{-1} holomorphic & derivatives of f & f^{-1} never vanish comes for free.

⑫ Examples of conformally equivalent domains: tie together $z \longmapsto z^2$ & möbius above. Beware of using $z \longmapsto z^2$ and forgetting $(0,\infty)$. "fans out"

Counter-examples: Domains not conf. equiv. to D.

① $\Omega = \mathbb{C}$, If $\varphi: \mathbb{C} \longrightarrow D$ biholo, then entire, bdd $\overset{\text{Louiville's thm}}{\Longrightarrow}$ φ constant $\Rightarrow$ not bijective

② Ω = Annulus not even $\underbrace{\text{homomorphic}}_{\text{topologically equiv}}$ to D.

connected but

not simply connected

⑬ Def 2.32: $\Omega \subset \mathbb{C}$ simply connected if Ω connected & every closed cts path $\gamma: [a,b] \longrightarrow \Omega$ is homotopic to the constant path $\tilde{\gamma}: [a,b] \longrightarrow \Omega$, $\tilde{\gamma}(t) = \gamma(a) = \gamma(b)$

↳ intuition: doesn't contain any holes.

note: $\underbrace{\gamma_1, \gamma_2}_{\text{same end pts}}$ homotopic if $\exists \overset{\text{cts}}{h}: [0,1] \times [a,b] \longrightarrow \Omega$ s.t.

① $\underbrace{\forall s \in [0,1],\ h(s,a) = \gamma_1(a),\ h(s,b) = \gamma_1(b)}_{\text{Same end points}}$

② $\underbrace{\forall t \in [a,b],\ h(0,t) = \gamma_1(t),\ h(1,t) = \gamma_2(t)}_{\text{interpolate from } \gamma_1 \text{ to } \gamma_2}$

Chapter 3 - Review of basic complex analysis II

(14) Power series thms:

- $(a_n) \in \mathbb{C}$, radius of convergence $R := \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \in [0,\infty]$, $\overbrace{\sum_{n=0}^{\infty} a_n z^n}^{f(z)}$ → converges for $|z|<R$ (root test); → diverges for $|z|>R$
- f is holo on $B_R(0)$, $f'(z)$ has same RoC. Term by term diff.
- f is infinitely diff & $f^{(n)}(0) = a_n n!$
- $\forall r \in (0,R)$, have uniform convergence: $\sum_{n=0}^{k} a_n z^n \xrightarrow{k\to\infty} \sum_{n=0}^{\infty} a_n z^n$ on $B_r(0)$

(15) Function defs:

- $\exp: \mathbb{C} \longrightarrow \mathbb{C}$ defined as $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\sinh(z) = \frac{e^z - e^{-z}}{2}$, $\cosh(z) = \frac{e^z + e^{-z}}{2}$, $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
- $z = |z|e^{i\theta}$, $\theta = \arg(z) \in \mathbb{R}/(2\pi\mathbb{Z})$. Beware! Branch cuts / principal values.
- $\log z = \log|z| + i\arg(z)$ (modulo $2\pi i$ ∵ arg multivalued)

Note: complex integration content assumed from analysis III

Recall: $\left|\int_\gamma f(z)dz\right| \leq \sup_\gamma |f| \cdot \text{length}(\gamma)$

(16) Lem 3.10:

$\Omega \subset \mathbb{C}$ open, $f: \Omega \longrightarrow \mathbb{C}$ cts, $F: \Omega \longrightarrow \mathbb{C}$ holo. $F'(z) = f(z)$. If γ pwC' closed curve in Ω, then $\int_\gamma f(z)dz = 0$

Pf: $\int_\gamma f(z)dz = \int_\gamma F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a)) = 0$ ∎

Chapter 4 - Winding Numbers

(17) Lem 4.1: (Lifting lemma)

$\gamma: [a,b] \longrightarrow \mathbb{C}\setminus\{0\}$ cts, fix $\theta_0 \in \mathbb{R}$ s.t. $\gamma(a) = |\gamma(a)|e^{i\theta_0}$. Then $\exists$ unique cts $\theta: [a,b] \longrightarrow \mathbb{R}$ s.t. $\theta(a) = \theta_0$ & $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ $\forall t \in [a,b]$

Pf: non-examb. Can now unambiguously define change in argument along cts path

(18) Def 4.2:

$\gamma: [a,b] \longrightarrow \mathbb{C}\setminus\{0\}$ cts, $\theta: [a,b] \longrightarrow \mathbb{R}$ lift from ↑, $\measuredangle(\gamma) = \theta(b) - \theta(a)$

note: no 2π ambiguity.

(19) Def 4.3:

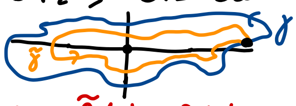
$\gamma: [a,b] \longrightarrow \mathbb{C}\setminus\{w\}$ closed cts path. Index / winding number of γ around w

$$I(\gamma, w) = \frac{1}{2\pi}\measuredangle(\gamma) \in \mathbb{Z}$$

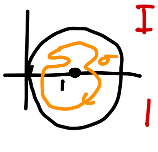
Small perturbations wind by the same amount

(20) Lem 4.6: (Dog walking)

$\gamma, \tilde{\gamma}: [a,b] \longrightarrow \mathbb{C}\setminus\{0\}$ cts closed paths. $|\underbrace{\gamma(t)}_{\text{you}} - \underbrace{\tilde{\gamma}(t)}_{\text{dog}}| < |\gamma(t)|$ (length of dog lead < dist to tree) $\forall t \in [a,b]$, then $I(\gamma, 0) = I(\tilde{\gamma}, 0)$



Pf: $\theta(t), \tilde{\theta}(t)$ lifts of $\arg(\gamma), \arg(\tilde{\gamma})$. $\alpha(t) := \tilde{\theta}(t) - \theta(t)$, $\sigma(t) := \frac{\tilde{\gamma}(t)}{\gamma(t)} = |\sigma(t)|e^{i\alpha(t)}$



$I(\tilde{\gamma}, 0) - I(\gamma, 0) = \underbrace{\frac{1}{2\pi}[\tilde{\theta}(b) - \tilde{\theta}(a)]}_{\text{definition}} - \underbrace{\frac{1}{2\pi}[\theta(b) - \theta(a)]}_{\text{definition}} = \frac{1}{2\pi}[\alpha(b) - \alpha(a)] = I(\sigma, 0)$

$|1 - \sigma(t)| = \left|\frac{\gamma(t) - \tilde{\gamma}(t)}{\gamma(t)}\right| < 1 \Rightarrow \sigma(t) \subset B_1(1) \overset{\text{avoids a radial line out to } \infty}{\Longrightarrow} I(\sigma, 0) = 0$ ∎

(21) Lem 4.7:

Winding number constant on each connected component of $\underbrace{\mathbb{C} \setminus \gamma([a,b])}_{\text{open in } \mathbb{C}}$ (cts image of compact set so compact → closed in $\mathbb{C}$)

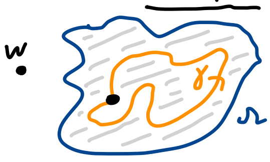
Pf: show $I(\gamma, \cdot)$ constant on nbhd of every pt.

(22) Thm 4.10:

$w \in \mathbb{C}$, $\gamma_0, \gamma_1: [a,b] \longrightarrow \mathbb{C}\setminus\{w\}$ homotopic cts closed paths. Then $I(\gamma_0, w) = I(\gamma_1, w)$

If γ homotopic to constant path $\Rightarrow I(\gamma, w) = 0$

↳ **Cor 4.11:** $\Omega \subset \mathbb{C}$ simply connected, then $\forall w \in \mathbb{C}\setminus\Omega$, $\gamma: [a,b] \longrightarrow \Omega$ cts closed, then $I(\gamma, w) = 0$ **Pf:** simply connected $\Rightarrow$ every path shrunk to const. path ∎



Pf: translation $\Rightarrow$ WLOG, $w = 0$. γ_1, γ_2 homotopic $\Rightarrow \exists h: [0,1] \times [a,b] \longrightarrow \mathbb{C}\setminus\{0\}$ (continuous) s.t. $\underbrace{h(0,t)=\gamma_0(t),\ h(1,t)=\gamma_1(t)}_{\text{interpolation}}\ \forall t\in[a,b]$, $\underbrace{h(s,a)=h(s,b)=z_0}_{\text{fixed end point}}\ \forall s\in[0,1]$ $\quad z_0=\gamma_0(a)=\gamma_0(b)=\gamma_1(a)=\gamma_1(b)$

- Define $\gamma_s:[a,b]\to\mathbb{C}\setminus\{0\}$ by $\gamma_s(t)=h(s,t)$. WTS $I(\gamma_s,0)$ constant in s.
- $[0,1]\times[a,b]$ compact, $\underbrace{|h|}_{\text{AIII}}$ cts $\Rightarrow$ achieves infimum $\varepsilon\geq 0$. We omit zero $\Rightarrow$ omit entire open ball $B_\varepsilon(0)$, $\varepsilon>0$
- h cts on compact domain $\Rightarrow$ uniformly cts $\Rightarrow$ $\exists\delta>0$ s.t. $\forall t\in[a,b]$, $\forall s_1,s_2\in[0,1]$ $(|s_1-s_2|<\delta)\Longrightarrow(|h(s_1,t)-h(s_2,t)|<\varepsilon)$
- $|\gamma_{s_1}(t)-\gamma_{s_2}(t)|=|h(s_1,t)-h(s_2,t)|<\underbrace{\varepsilon\leq|\gamma_{s_1}(t)|}_{\text{outside }B_\varepsilon(0)}\xRightarrow{(20)}\underbrace{I(\gamma_{s_1},0)=I(\gamma_{s_2},0)}_{\text{winding \# equal}}$
  dog walking
  $s\mapsto I(\gamma_s,0)$ constant ■

(23) **Lem 4.12:** $w\in\mathbb{C}$, $\gamma:[a,b]\to\mathbb{C}\setminus\{w\}$ closed pw C^1, Then

$$\boxed{I(\gamma,w)=\frac{1}{2\pi i}\int_\gamma\frac{dz}{z-w}}$$

Winding # as an integral

Pf: Translation $\Rightarrow$ $w=0$ WLOG, assume not piecewise.

$$\int_\gamma\frac{dz}{z}=\int_a^b\frac{\gamma'(t)}{\gamma(t)}dt.$$ Take $\theta(t)$ as lift $\Rightarrow$ $\gamma(t)=\underbrace{|\gamma(t)|e^{i\theta(t)}}_{(*)}$ is C^1

School calc $\Rightarrow$ $\gamma'(t)=e^{i\theta(t)}\frac{d}{dt}|\gamma(t)|+|\gamma(t)|\,i\,\theta'(t)e^{i\theta(t)}$

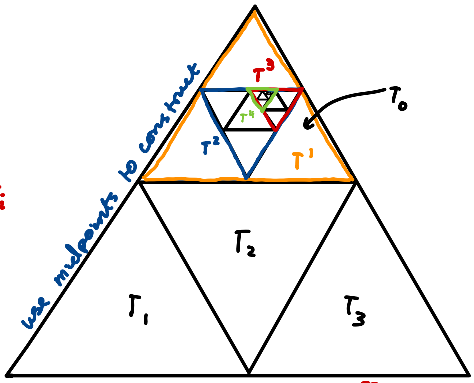
But $$\frac{\gamma'(t)}{\gamma(t)}=\frac{d}{dt}\log|\gamma(t)|+i\theta'(t)$$ [simply divide by *]

$$\int_\gamma\frac{dz}{z}=\int_a^b\left[\frac{d}{dt}\log|\gamma(t)|+i\theta'(t)\right]dt=0+i[\theta(b)-\theta(a)]=i\measuredangle(\gamma)=2\pi i\overbrace{I(\gamma,0)}^{\text{winding \#}}$$ ■

Chapter 5 - Cauchy's Theorem

(24) **Thm 5.3:** (Goursat's thm) $\Omega\subset\mathbb{C}$ open, closed triangle $T\subset\Omega$. $f:\Omega\to\mathbb{C}$ holo. Then

$$\int_{\partial T}f(z)dz=0$$



Pf: Divide T into T_0,T_1,T_2,T_3. By edge cancelation, $\int_{\partial T}=\sum_{i=0}^3\int_{\partial T_i}$

Δ ineq & pick largest: $\left|\int_{\partial T}f(z)dz\right|\leq\sum_{i=1}^3\left|\int_{\partial T_i}f(z)dz\right|\leq 4\left|\int_{\underbrace{\partial T'}_{\text{pick }T\text{ from }T_0,T_1,T_2,T_3}}f(z)dz\right|$

Iterate, $T_2\subset T_1$ etc: nested sequence T^n w/ geom decay
$\text{diam}(T^n)=2^{-n}\text{diam}(T)$, $L(\partial T^n)=2^{-n}L(\partial T)$ [length]
and $\left|\int_{\partial T}f(z)dz\right|\leq 4^n\left|\int_{\partial T^n}f(z)dz\right|$, $\forall n\in\mathbb{N}$, pick $z_n\in T^n$. Diam $\to 0$ $\Rightarrow$ z_n cauchy $\Rightarrow$ $z_n=\bigcap_{i=0}^\infty T^n\in T\subset\Omega$
$|R(z)|\leq\varepsilon|z-z_\infty|$

Apply complex diff at z_∞: $\forall\varepsilon>0$ $\exists\delta>0$ s.t. $\forall z\in B_\delta(z_\infty)$, $f(z)=f(z_\infty)+(z-z_\infty)f'(z_\infty)+R(z)$

$\underbrace{T^n\in B_\delta(z_\infty)}_{\text{for large enough }n}$: $\int_{\partial T^n}f(z)dz=\int_{\partial T^n}[f(z_\infty)+(z-z_\infty)f'(z_\infty)+R(z)]dz$

$$\boxed{\int_{\partial T^n}z^n dz=0\quad\text{for } n\neq -1}$$

$$=(f(z_\infty)-f'(z_\infty)z_\infty)\cancel{\int_{\partial T^n}dz}+f'(z_\infty)\cancel{\int_{\partial T^n}z\,dz}+\int_{\partial T^n}R(z)dz$$

Controlling $\left|\int_{\partial T^n}f(z)dz\right|\leq\overbrace{\varepsilon\underbrace{\sup_{z\in\partial T^n}|z-z_\infty|}_{\leq\text{diam}(T^n)}}^{\text{max value integrand}}\cdot\underbrace{2^{-n}L(\partial T)}_{\leq\text{length }\partial T^n}\leq\varepsilon\cdot 4^{-n}\cdot[\text{diam}(T)\cdot L(\partial T)]$

fixed

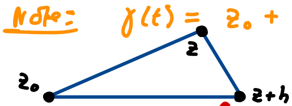
$\Rightarrow\left|\int_{\partial T}f(z)dz\right|\leq 4^n\cdot\varepsilon\cdot 4^{-n}\cdot[\text{diam}(T)\cdot L(\partial T)]=C\varepsilon\xrightarrow[\text{arbitrary}]{\varepsilon\to 0}0$ ■

(25) **Def 5.4:** $\Omega\subset\mathbb{C}$ open is a star shaped domain if $\exists z_0\in\Omega$ s.t. $\forall z\in\Omega$ $\underbrace{[z_0,z]}_{\text{line segment}}\subset\Omega$. z_0 is called a central point.

$\boxed{\text{Goursat's lets us construct anti derivatives for cts functions on sufficiently nice domains}}$

(26) **Thm 5.5:** $\Omega \subset \mathbb{C}$ star shaped, $f: \Omega \to \mathbb{C}$ cts. Spps $\forall$ Triangles $T \subset \Omega$, $\int_{\partial T} f(z)\,dz = 0$, then $\exists$ holomorphic $F: \Omega \to \mathbb{C}$ s.t. $F'(z) = f(z)$. If z_0 is the central point of Ω, then $F(z) = \int_{[z, z_0]} f(z)\,dz$ works.

Note: $\gamma(t) = z_0 + t(z - z_0)$ for $t \in [0,1]$ w/ $\gamma = [z_0, z]$



write 3 separate integrals

Pf: Fix $z \in \Omega$, pick $r > 0$ s.t. $B_r(z) \subset \Omega$. Triangle ∂, know $\int_{\partial T} f(z)\,dz = 0 = \underbrace{F(z+h)}_{[z+h,\, z_0]} - \underbrace{\int_{[z, z+h]} f(w)\,dw}_{[z,\, z+h]} - \underbrace{F(z)}_{[z_0,\, z]}$

Also, $\int_{[z, z+h]} dw = \int_0^1 \gamma'(t)\,dt = \gamma(1) - \gamma(0) = h$, note that $f(z)$ is NOT a function of w, so

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{[z, z+h]} \big(f(w) - f(z)\big)\,dw \right| \le \max_{w \in [z, z+h]} |f(w) - f(z)| \longrightarrow 0 \quad \blacksquare$$

↳ **Cor 5.6:** take f holo, then $\exists F$ holo s.t. $F' = f$, $F(z) = \int_{[z_0, z]} f(w)\,dw$.

(27) **Thm 5.7:** (Cauchy's Thm on Star shaped domains). Ω star shaped, f holo, $\gamma \in C^1$pw closed $\Rightarrow \int_\gamma f(z)\,dz = 0$

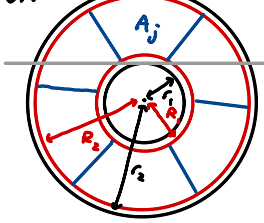
Pf: Goursat's $\Rightarrow \exists F$ holo w/ $F' = f$ & apply FTC & chain rule as in (16) $\blacksquare$

(28) **Cor 5.8:** (Cauchy's Thm on Annuli). f holo on $A_{r_1, r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, then $\int_{\partial A} f(z)\,dz = 0$

$r_1 < R_1 < R_2 < r_2$

Pf: construct a star shaped domain by subdividing A until $\forall j$ $\int_{\partial A_j} f(z)\,dz = 0$

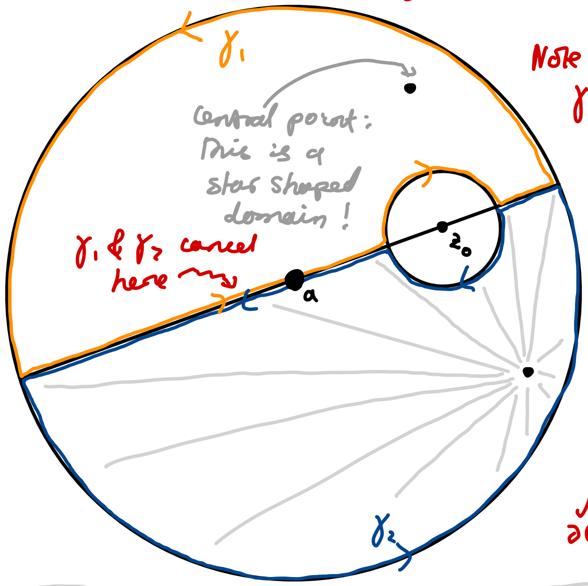
↳ Idea: break up domains until you achieve star shaped! (Q5.3)



(29) **Thm 5.9:** (Cauchy's integral formula on a disk). $\Omega \subset \mathbb{C}$ open, $f: \Omega \to \mathbb{C}$ holo. $\overline{B_r(z_0)} \subset \Omega$, then $\forall z_0 \in B_r(a)$, we have

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(z)}{z - z_0}\,dz}$$

Pf: Pick $\delta > 0$ s.t. $B_\delta(z_0) \subset B_r(a)$



Note that $z \longmapsto \frac{f(z) - f(z_0)}{z - z_0}$ is holo on $\Omega \setminus \{z_0\}$, interiors of γ_1 and γ_2 are star shaped so for $k \in \{1, 2\}$ by Cauchy

$$\int_{\gamma_k} \frac{f(z) - f(z_0)}{z - z_0}\,dz = 0 \overset{\text{adding}}{\Longrightarrow} \int_{\partial B_r(a)} \frac{f(z) - f(z_0)}{z - z_0}\,dz \overset{(*)}{=} \int_{\partial B_\delta(z_0)} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

As $z \to z_0$, $\frac{f(z) - f(z_0)}{z - z_0} \longrightarrow f'(z_0)$

$$\Rightarrow |\text{RHS } *| \le \underbrace{|f'(z_0) + 1|}_{\max} \cdot \underbrace{2\pi\delta}_{\text{length}} \xrightarrow{\delta \to 0} 0$$

So rearranging,

$$\int_{\partial B_r(a)} \frac{f(z)}{z - z_0}\,dz = \int_{\partial B_r(a)} \frac{f(z_0)}{z - z_0}\,dz = f(z_0)\, 2\pi i\, \overbrace{I(\partial B_r(a), a)}^{\text{def winding \#}} = 2\pi i\, f(z_0)$$

$\blacksquare$

Chapter 6 – Taylor series & Applications

(30) **Thm 6.1:** $\Omega \subset \mathbb{C}$ open, $f: \Omega \to \mathbb{C}$ holo. $B_r(z_0) \subset \Omega$, then $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ $\forall z \in B_r(z_0)$ w/ $a_k = \frac{1}{2\pi i} \int_{\partial B_\delta(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\,dw$ note: $\delta \in (0, r]$ by Cauchy's Thm on Annuli

↳ **Cor 6.4:** f is infinitely differentiable Pf: (14) $\blacksquare$

↳ **Cor 6.5:** $\forall n \in \mathbb{N}$, can say that $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}\,dw$ Pf: $\frac{f^{(n)}(z_0)}{n!} = a_n$ $\blacksquare$

Note: Says that $\underbrace{f \text{ holo}}_{\text{complex diff } \forall z \in \Omega} \Rightarrow \underbrace{f \text{ analytic}}_{\text{expand as pwr series around ball}}$, (14) says analytic $\Rightarrow$ holo. Equivalence from Theory!

Pf: Translation $\Rightarrow z_0 = 0$ $\quad \frac{1}{w - z} = \frac{1}{w}\left(\frac{1}{1 - \frac{z}{w}}\right) = \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k$, $|z| < |w| = r$

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(w)}{w - z}\,dw = \frac{1}{2\pi i} \int_{\partial B_r(0)} f(w) \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}\,dw \underset{\text{JU} \therefore\,?}{=} \frac{1}{2\pi i} \sum_{k=0}^{\infty} \Bigg(\overbrace{\int_{\partial B_r(0)} \frac{f(w)}{w^{k+1}}\,dw}^{2\pi i\, a_k} \Bigg) z^k \quad \blacksquare$$

↳ Cor 6.6: $\overbrace{|f(z)|<M}^{\text{control of } f} \Rightarrow \overbrace{|a_k| < \frac{M}{R^k}}^{\text{control of } a_k}$ Pf: $|a_k| \le \frac{1}{2\pi}\left|\int_{\partial B_r(0)} \frac{f(w)}{w^{k+1}}\,dw\right| \le \frac{1}{2\pi}\cdot\overbrace{\frac{M}{r^{k+1}}}^{\text{max value}}\cdot\underbrace{2\pi r}_{\text{length}} \xrightarrow{r\to R} \frac{M}{R^k}$ ∎

↳ Cor 6.7: (Liouville's Thm) bbd, entire ⇒ constant Pf: write f as taylor series, $|a_k| \le \frac{M}{R^k} \xrightarrow[k\ge 1]{R\to\infty} 0$ ∎

(31) Thm 6.9: (Morera's thm) $\Omega \subset \mathbb{C}$ open, $f: \Omega \to \mathbb{C}$ cts, $\forall$ closed triangles T, $\overbrace{\int_{\partial T} f(z)\,dz = 0}^{(*)}$, then, f is holo on Ω. Note: f just cts, $(*) \Rightarrow$ inf diff! Amazing!!

Pf: WTS f holo at $a \in \Omega$, pick $B_r(a) \subset \Omega$, (26) $\Rightarrow \exists$ holo $F: B_r(a) \to \mathbb{C}$ s.t. $F' = f$.
F holo $\Rightarrow$ F ∞ diff, in particular, f complex diff at a. ∎

(32) Lem 6.11: (holo. f locally invertable where $f' \neq 0$). If $f'(z_0) \neq 0$, $\exists$ nbhds $V_0 \subset \Omega$ of z_0, $V_1 \subset \mathbb{C}$ of $f(z_0)$ s.t. $f|_{V_0}$ is biholo $V_0 \longrightarrow V_1$ Pf: inverse function thm from MVC ∎

Chapter 7 - Zeros of holomorphic functions

Revision Lecture Says very important

(33) Def 7.1: $\Omega \subset \mathbb{C}$, $f: \Omega \to \mathbb{C}$ holo w/ $f(z_0)=0$, Then order of the zero of f at z_0

$$\operatorname{ord}(f, z_0) = \begin{cases} \infty & \text{if } f^{(k)}(z_0) = 0 \quad \forall k \in \mathbb{N} \\ \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\} & \text{o/w} \end{cases}$$

(34) Thm 7.3: $f: \Omega \to \mathbb{C}$ holo has $\operatorname{ord}(f, z_0) = n$, $z_0 \in \Omega$, then $\exists g: \Omega \to \mathbb{C}$ holo s.t. $f(z) = (z - z_0)^n g(z)$, $g(z_0) \neq 0$ in some neighbourhood of z_0.
Pf: $f(z) = \sum_{k=0}^{\infty} a_k z^k$. $a_k = \frac{f^{(k)}(z_0)}{k!} \Rightarrow a_k = 0 \ \forall k < n \Rightarrow f(z) = (z-z_0)^n \underbrace{\sum_{k=0}^{\infty} a_{k+n}(z-z_0)^k}_{g(z)}$
"The higher the order, the flatter the function!"

(35) Thm 7.4: Ω open & connected, $f: \Omega \to \mathbb{C}$ holo w/ zero of ∞ order at z_0 $\Rightarrow f \equiv 0$

Pf: Consider $\Omega_0 = \{z \in \Omega : f$ has a zero of inf order at $z\}$. WTS $\Omega = \Omega_0$. as $\Omega_0 \neq \emptyset$
WTS Ω_0 both open & closed
know $\Omega_0 \neq \emptyset$ $\because z_0 \in \Omega_0$. Ω connected $\Rightarrow$ only $\underbrace{\emptyset}_{\text{empty set}}$ & $\underbrace{\Omega}_{\text{whole set}}$ both open & closed

- Open: Pick $w \in \Omega_0$, write $f(z) = \sum_{k=0}^{\infty} a_k (z-w)^k$, but $a_k = \frac{f^{(k)}(w)}{k!} = 0 \Rightarrow f \equiv 0$ on $B_r(w) \subset \Omega$
$\Rightarrow B_r(w) \subset \Omega_0 \Rightarrow \Omega_0$ open
- closed: Sequence $z_i \in \Omega_0$ w/ $z_i \longrightarrow z_\infty \in \Omega$. By cty, $f(z_\infty) = 0$. If z_∞ was a zero w/ finite order $\Rightarrow \underbrace{\text{isolated zero } \times}_{\text{but } z_i \longrightarrow z_\infty} \Rightarrow z_\infty \in \Omega_0 \Rightarrow \Omega_0$ closed ∎

(36) Thm 7.5: (Identity Thm). $\Omega \subset \mathbb{C}$ open & connected, $f_0, f_1 : \Omega \longrightarrow \mathbb{C}$ holo. $\Sigma = \{z \in \Omega : f_0(z) = f_1(z)\}$ has an accumulation pt in Ω. Then $f_1 \equiv f_2$
z_∞ acc pt of Σ if $\exists z_i \in \Sigma \setminus \{z_\infty\}$ s.t. $z_i \longrightarrow z_\infty$
↳ $\Longleftrightarrow$ two holo fctns on open connected Ω are either ① identical, ② agree at isolated pts

Pf: $g = f_1 - f_0$ holo & $g = 0$ on Σ, accumulation pt has $g(z_i) = 0 \Rightarrow g(z_\infty) = 0$
$\Rightarrow$ NOT an isolated zero & infinite order $\Rightarrow g \equiv 0 \Longleftrightarrow f_1 \equiv f_2$ ∎

(37) Thm 7.8: $f: \Omega \longrightarrow \mathbb{C}$ holo, if $\operatorname{ord}(f, z_0) = k \geq 1$. Then $\exists$ nbhd $V_0 \subset \Omega$ of z_0, $r > 0$, & $h: V_0 \longrightarrow B_r(0)$ s.t. $\forall z \in V_0$ $f(z) = (h(z))^k$ Pf: skipped but examb...
↳ f locally k-to-one near z_0 [think injections!]

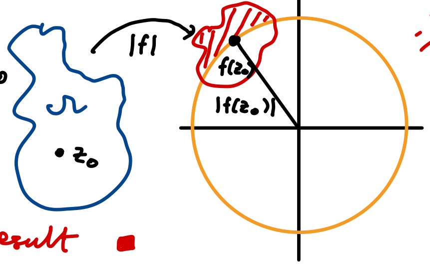
(38) Lem 7.9: $g: \Omega \to \mathbb{C} \setminus \{0\}$ holo & $\exists F$ holo s.t. $F' = \frac{g'}{g} \Rightarrow \exists w$ s.t. $g(z) = e^{\ell(z)}$ $\forall z \in \Omega$ $\ell(z) = F(z) + w_0$
Pf: know this - actually "fair game"!
↳ Cor 7.10: Ω star shaped, $g: \Omega \longrightarrow \mathbb{C} \setminus \{0\}$ holo, then $\exists \ell$ holo s.t. $g(z) = e^{\ell(z)}$ unique up to $2\pi i \mathbb{Z}$

(39) Thm 7.11: (Open Mapping Thm). Ω open, connected, $f: \Omega \longrightarrow \mathbb{C}$ holo but not constant.
Then $f(\Omega)$ open & connected

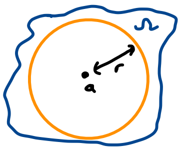
Pf: Topology: image of connected set under cts map still connected. WTS open.
Open: pick $w_0 = f(z_0) \in f(\Omega)$. WTS $f(\Omega)$ contains whole nbhd of w_0
Set $g(z) = f(z) - w_0$. $\operatorname{ord}(g, z_0) = k < \infty$. [If ord zero $\infty \Rightarrow g \equiv 0$ constant ✗]
(37) $\Rightarrow$ $f(z) = w_0 + (h(z))^k$ w/ $h: V_0 \longrightarrow B_\epsilon(0)$ biholo
h^k maps onto $B_{\epsilon^k}(0) \Rightarrow f$ maps onto $B_{\epsilon^k}(w_0)$ [nbhd of w_0] ■

↳ Cor 7.12: (Maximum Modulus Principal). Ω open, connected. $f: \Omega \longrightarrow \mathbb{C}$ holo, not constant $\Longrightarrow$ f doesn't have any local maxima.

Pf: Suppose $|f|$ achieves max at $z_0 \in \Omega$. OMT $\Rightarrow f(\Omega)$ open $\Rightarrow$ contains nbhd of $f(z_0)$
↳ But some must spill out of $B_{|f(z_0)|}(0) \Longrightarrow |f|$ doesn't achieve max at z_0 ✗ ■



(40) Lem 7.13: (Mean Value Property) $\overline{B_r(z_0)} \subset \Omega$, $f: \Omega \longrightarrow \mathbb{C}$ holo
Then



$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Pf: Cauchy integral formula: $f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{w - z_0}\, dw$ = param = result ■

(41) Thm 7.14: Ω domain, $f: \Omega \longrightarrow \mathbb{C}$ injective & holo. Then $f(\Omega)$ domain & $f: \Omega \longrightarrow f(\Omega)$ is a biholomorphic map

note f injective $\Rightarrow f$ not constant

Pf: Ω connected, f cts $\Rightarrow f(\Omega)$ connected. OMT (39) $\Rightarrow f(\Omega)$ open $\Longrightarrow f(\Omega)$ domain in $\mathbb{C}$

zero of order $k \geq 2$!
Sppse $\exists z_0 \in \Omega$ s.t. $f'(z_0) = 0$. Then $F(z) = f(z) - f(z_0) \Rightarrow F(z_0) = F'(z_0) = 0$
(37) $\Longrightarrow$ F not injective $\because$ Locally 2-1 near z_0. $\Longrightarrow f$ not injective ✗.

$f'(z) \neq 0 \ \forall z \in \Omega$, (32) $\Longrightarrow f$ local biholo. f bijective $\Rightarrow$ global biholo ■

(42) Thm 7.15: (Schwartz Lemma) $f: D \longrightarrow D$ holo on D w/ $f(0) = 0$. Then

(i) $|f'(0)| \leq 1$
(ii) $|f(z)| \leq |z| \quad \forall z \in D$

If equality holds anywhere on $D \setminus \{0\}$, then f is a rotation: $f(z) = e^{i\theta} z$, $\theta \in \mathbb{R}$

call this g
$z(z^{n-1} g(z))$

Pf: zero at $z=0$ finite (o/w $f \equiv 0$ & thm trivial) $\Rightarrow \exists g: D \longrightarrow \mathbb{C}$ holo s.t. $f(z) = z\,g(z)$

need smaller disk for boundary
open unit disk
$|z|$
For $r \in (0,1)$, $\forall z$ w/ $|z| = r$, $1 > |f(z)| = r|g(z)| \Rightarrow |g(z)| \leq \frac{1}{r}$

(39) Max modulus principal: $|g|$ attains max over $B_r(0)$ on boundary $\{|z| = r\}$

on D $\quad f'(z) = g(z) + z \cdot g'(0)$
$\Rightarrow |g(z)| < \frac{1}{r} \ \forall |z| < r \xrightarrow{r \uparrow 1} |g(z)| \leq 1 \Rightarrow |f'(0)| \leq 1$, $|f(z)| \leq |z|$
$|f(z)| = |z||g(z)|$

Equality $\Rightarrow |g(z_0)| = 1$ for $z_0 \in D \setminus \{0\} \Longrightarrow g$ attains local max at z_0. Maximum modulus principal $\Rightarrow |g(z)| = 1 \Rightarrow g(z) = e^{i\theta}$ ■

↳ Cor 7.16: Every biholo $f: D \longrightarrow D$ is möbius, $f(z) = e^{i\theta}\left(\frac{z - a}{\bar{a}z - 1}\right)$ $|a| < 1$, $\theta \in [-\pi, \pi)$

apply to f^{-1}
Pf: If f biholo has $f(0) = 0$, Schwartz (42) $\Rightarrow |f(z)| \leq |z|$ & $|z| \leq |f(z)|$. $\Rightarrow |f(z)| = |z|$
If $f(0) \neq 0$, $a = f^{-1}(0)$, $\varphi(z) = \frac{z - a}{\bar{a}z - 1}$ biholo $D \longrightarrow D$ w/ $\varphi(a) = 0 \Rightarrow f \circ \varphi$ maps $0 \mapsto 0$ ■
rotation too...

Chapter 8 - Isolated Singularities

(43) Def 8.1: $f: B_r(a) \setminus \{a\} \longrightarrow \mathbb{C}$ holo, $r > 0$, $a \in \mathbb{C}$, has an isolated singularity at a.

(44) **Thm 8.2:** (Riemann's removable singularity Thm). $f: B_r(a)\setminus\{a\} \longrightarrow \mathbb{C}$ holo. Suppose $\lim_{z\to a}(z-a)f(z) = 0$. Then f extends to holo function $f: B_r(a) \longrightarrow \mathbb{C}$

Pf: Define $g: B_r(a) \longrightarrow \mathbb{C}$ by $g(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \in B_r(a)\setminus\{a\} \\ 0 & \text{if } z = a \end{cases}$

$g(z)$ complex diff away from a by product rule. $\boxed{\text{WTS diff at } \{a\}}$. $\dfrac{g(z) - \overset{=0}{g(a)}}{z-a} = \dfrac{(z-a)^2 f(z)}{z-a} = (z-a)f(z) \xrightarrow{z\to a} 0$

$\Rightarrow g$ complex diff at a w/ $g'(a) = 0 \implies$ zero of order at least two... $\boxed{\text{How to extend?}}$

$\Rightarrow \operatorname{ord}(g,a) = n < \infty$ [o/w $g \equiv 0 \Rightarrow f \equiv 0$ ✗] $\Rightarrow g(z) = (z-a)^n h(z) \rightsquigarrow f(z) = (z-a)^{n-2} h(z)$ ∎

cor: $f: \Omega \to \mathbb{C}$ holo, $z \longmapsto \dfrac{f(z)-f(w)}{z-w}$ extends to all of Ω, including $\{w\}$

(45) **Def 8.3:** (classification of isolated singularities) $f: B_r(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$ has a

(i) removable singularity at z_0 if $\underbrace{\lim_{z\to z_0} f(z)}_{\text{'has a limit'}} \in \mathbb{C} \setminus \{\infty\}$

(ii) pole at z_0 if $\lim_{z\to z_0} f(z) = \infty$ [just mapping to pt on Riemann sphere - not scary]

(iii) essential singularity at z_0 if neither (i) or (ii) hold.

(34) analogue

(46) **Thm 8.4:** $f: B_r(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$, pole at z_0. Then $\exists n \in \mathbb{N}$ (1, 2, 3, ...), $g: B_r(z_0) \longrightarrow \mathbb{C}$ holo (non-zero in nbhd of z_0) s.t. $f(z) = (z-z_0)^{-n} g(z)$ — $f \to \infty$ so can do this!

Riemann removable singularity thm: $\exists F: B_r(z_0) \to \mathbb{C}$, $F(z_0) = 0$, $F(z) = \frac{1}{f(z)}$

Pf: reduce $r > 0$ s.t. $\underbrace{|f(z)| \geq 1}$ $\forall z \in B_r(z_0)\setminus\{z_0\} \Rightarrow \underbrace{\frac{1}{f} \text{ bbd, holo on } B_r(z_0)\setminus\{z_0\}}$

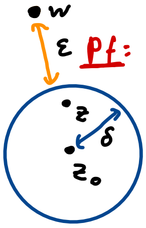
zero of F finite [o/w $F \equiv 0$ but $F(z) = \frac{1}{f(z)} \neq 0$] $\Rightarrow \operatorname{ord}(F, z_0) = n$, $F(z) = (z-z_0)^n \underbrace{G(z)}_{\frac{1}{g(z)}}$ ∎

(47) **Def 8.5:** Ω open connected, $f: \Omega \longrightarrow \mathbb{C}_\infty$ is cts, $f \not\equiv \infty$. f meromorphic if f complex diff $\forall z_0 \in \Omega$, $f(z_0) \neq \infty$ and $\frac{1}{f}$ complex diff $\forall z_0 \in \Omega$ w/ $f(z_0) = \infty$ $\quad \underbrace{Z := f^{-1}(0)}_{\text{set of zeros}}$, $\underbrace{P := f^{-1}(\infty)}_{\text{set of 'poles'}}$ Both P, Z closed ∵ f cts; Z discrete, $\Omega\setminus Z$ connected

↓ ↓ CRAZY!!! ↓ ↓

(48) **Thm 8.6:** (Casorati-Weierstrauss thm) $f: B_r(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$ holo has an essential singularity at z_0. Then however small we take $\delta \in (0,r)$, $f(B_\delta(z_0)\setminus\{z_0\})$ dense in $\mathbb{C}$

(essential) $\Rightarrow$ (dense) — function is 'onto', regardless of how small domain



Pf: Prove contrapositive: ($\neg$ dense) $\Rightarrow$ ($\neg$ essential), so $\exists \delta \in (0,r)$, $\varepsilon > 0$, $w \in \mathbb{C}$ s.t.

$|f(z) - w| \geq \varepsilon \quad \forall z \in B_\delta(z_0)\setminus\{z_0\}$ [$\exists$ some point that the image doesn't hit]

WTS f has either ① removable singularity or ② a pole. Set $h(z) = \frac{1}{f(z)-w}$

$|h| \leq \frac{1}{\varepsilon}$ bbd, $h \neq 0$ on $B_\delta(z_0)\setminus\{z_0\}$. RRST (44) $\Rightarrow h: B_\delta(z_0) \longrightarrow \mathbb{C}$ [extended]

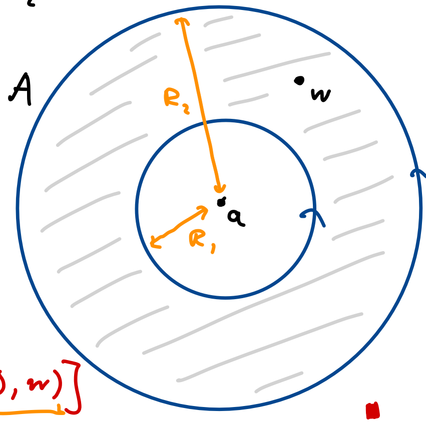
① If $h(z_0) \neq 0 \Rightarrow f(z) = \frac{1}{h(z)} + w \Rightarrow f$ has holo extension to $B_\delta(z_0) \Rightarrow$ removable singularity

② If $h(z_0) = 0 \Rightarrow \underbrace{\operatorname{ord}(h, z_0) = n < \infty}_{\text{o/w } h \equiv 0 \text{ by (35)}} \Rightarrow h(z) = (z-z_0)^n g(z) \Rightarrow f(z) = \underbrace{(z-z_0)^{-n}\frac{1}{g(z)} + w}_{\text{pole at } z_0}$

$g: B_\delta(z_0) \to \mathbb{C}$ non-zero at z_0 ∎

(49) **Thm 8.9:** (Cauchy Integral Formula for Annuli) $f: \Omega \to \mathbb{C}$ holo, $A = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$ (✱) $\bar{A} \subset \Omega$, then

$$f(w) = \frac{1}{2\pi i}\int_{\partial A} \frac{f(z)}{z-w}\,dz$$



fix $w \in A$

Pf: $z \longmapsto \frac{f(z)-f(w)}{z-w}$ holo on Ω. (extends! (44))

[Cauchy's thm for Annuli]

$$\Rightarrow \int_{\partial A} \frac{f(z)-f(w)}{z-w}\,dz = 0 \Rightarrow \int_{\partial A} \frac{f(z)}{z-w}\,dz = \underbrace{\int_{\partial A} \frac{f(w)}{z-w}\,dw}_{(✱)}$$

$$(✱) = f(w)\left(\int_{\partial B_{R_2}(a)} \frac{1}{z-w}\,dz - \int_{\partial B_{R_1}(a)} \frac{1}{z-w}\,dz\right) = 2\pi i\left[\overbrace{I(\partial B_{R_2}(a), w)}^{=1} - \underbrace{I(\partial B_{R_1}(a), w)}_{=0}\right]$$ ∎

(50) **Thm 8.12:** f holo on A (✱), then $\forall z \in A$, we have

$$f(z) = \sum_{k\in\mathbb{Z}} a_k (z-a)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i}\int_{\partial B_s(a)} \frac{f(w)}{(w-a)^{k+1}}\,dw \quad \forall k \in \mathbb{Z},\ \forall s \in (r_1, r_2)$$

$\operatorname{ord}(f,a) = \inf\{n \in \mathbb{Z} : a_n \neq 0\}$

Pf:  Similar to Taylors +  reciprocal expansion $\Rightarrow$  ∎

$|z| < R_2$ $\qquad$ $|z| > R_1$

(51) **Thm 8.17:** If f is injective & entire, then $f(z) = \alpha z + \beta$ w/ $\alpha \in \mathbb{C}\setminus\{0\}$, $\beta \in \mathbb{C}$

Pf: $g: \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$, $g(z) = f(\frac{1}{z})$ injective & holo $\because$ $z \mapsto \frac{1}{z}$ & f inj. & holo.
WTS g has a pole at zero $\Rightarrow$ rule out other possibilities

① If removable, limit exists $\Rightarrow$ g bdd in $\overline{D}\setminus\{0\}$ $\Rightarrow$ f bdd on $\mathbb{C}\setminus D$. f cts $\Rightarrow$ bdd on D
$\Rightarrow$ f bdd on $\mathbb{C}$. Entire so Liouville (30) $\Rightarrow$ f constant [but f injective] $\times$

② If essential, Casorati-Weierstrauss $\Rightarrow$ $g(D\setminus\{0\})$ dense $\Rightarrow$ $f(\mathbb{C}\setminus\overline{D})$ dense (in $\mathbb{C}$)
$f(D)$ open by OMT (39) $\Rightarrow$ $f(D) \cap f(\mathbb{C}\setminus\overline{D}) \neq \phi$. But f injective $\times$

$\Rightarrow$ 0 is a pole for g $\Rightarrow$ if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \overbrace{\sum_{k=-\infty}^{0} a_{-k} z^k}^{f(\frac{1}{z})}$ has a pole [say order n]
so $a_k = 0$ $\forall k > n$ $\Rightarrow$ f polynomial! But the only injective polynomials are degree 1 ∎

note: If cts, can extend holo functions on lines, will ignore Schwartz reflection principal...

Chapter 9 – The General Form of Cauchy's Theorem

note: $\underbrace{\gamma := \sum_{k=1} \alpha_k \gamma_k}_{\text{chain}}$ {A cycle is a chain w/ each γ_k a closed pw C' curve}

(52) **Def 9.1:** A cycle γ in Ω is homologous to zero in Ω if for any $a \in \mathbb{C}\setminus\Omega$, $I(\gamma, a) = 0$

(53) **Thm 9.3:** (Homology version - Cauchy's thm). Let $\Omega \subset \mathbb{C}$ be open, $f: \Omega \longrightarrow \mathbb{C}$ holo. Then for any cycle γ homologous to zero

$$\boxed{\int_\gamma f(z)\,dz = 0}$$

(proof of LNs...)
Pf: very technical so won't come up in exam? steps: ① pull in ② Cauchy integral formula ③ spot winding number ∎

↳ **Cor 9.4:** (Cauchy's integral formula). GENERAL VERSION. $\Omega \subset \mathbb{C}$ open, γ cycle (closed pw C' curve) in Ω, homologous to zero in Ω,
Then for $f: \Omega \longrightarrow \mathbb{C}$ holo, $\forall w \in \Omega\setminus\gamma([a,b])$

$$\boxed{f(w)\,I(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz}$$

Pf: (44) $\Rightarrow$ $z \longmapsto \frac{f(z)-f(w)}{z-w}$ extends from $\Omega\setminus\{w\}$ to Ω, Cauchy $\Rightarrow$ $\int_\gamma g(z)\,dz = 0$
Hence,

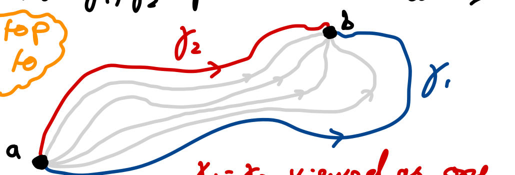
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dz = f(w)\,\underbrace{I(\gamma, w)}_{\text{winding number!}}$$ ∎

[might need Ω simply connected...] deformed to a point

(54) **Thm 9.7:** (Deformation thm) $\Omega \subset \mathbb{C}$ open, $f: \Omega \longrightarrow \mathbb{C}$ holo. If γ_1, γ_2 piecewise C' curves that are homotopic, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

{No holes to stop γ_1 deforming to γ_2}



$\gamma_1 - \gamma_2$ viewed as one closed curve

Pf: ... no need...

(55) **Def 9.9:** $f: B_\delta(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$ holo. Residue $\operatorname{res}(f, z_0) = \frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} f(z)\,dz$ $\forall \varepsilon \in (0, \delta)$

(56) **Thm 9.10:** (Residue Thm). Let $\Omega \subset \mathbb{C}$ open, f holo on $\Omega\setminus S$ (discrete set closed in Ω), γ closed pw C' in $\Omega\setminus S$, that is homologous to zero in Ω. Then $\exists$ finitely many $a \in S$ s.t. $I(\gamma, a) \neq 0$ &

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{a\in S} I(\gamma, a)\operatorname{res}(f, a)}$$

highly doubt proof will come up...

Pf: set $A = \{a \in S : I(\gamma, a) \neq 0\}$, WTS A is finite, suppose its not for contradiction, (!)

57 Ex 9.6: (Evaluating residues) Consider f with a 'problem' at z_0

(a) Removable singularities: then $\lim_{z\to z_0} f(z)$ exists, Cauchy's thm $\Rightarrow$ $\underbrace{\boxed{res(f,z_0)=0}}$ removable

(b) Simple poles: write $f(z)=\frac{g(z)}{z-z_0}$, $res(f,z_0) = \overbrace{\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} \frac{g(z)}{z-z_0}dz}^{\text{definition of res}} = \boxed{\underbrace{g(z_0)}_{\text{Cauchy's integral formula}} = res(f,z_0)}$ simple poles

(c) Ratios w/ simple pole: $f(z)=\frac{h(z)}{k(z)}$, $\boxed{res(f,z_0) = \frac{h(z_0)}{k'(z_0)}}$ ratio w/ simple pole

Pf: want $f(z) = \frac{g(z)}{z-z_0}$, so let $g(z) = \frac{h(z)}{\left(\frac{k(z)-k(z_0)}{z-z_0}\right)}$ derivative $\xrightarrow{z\to z_0} \frac{h(z_0)}{k'(z_0)} = g(z_0) = res(f,z_0)$

$k(z_0)=0$ $\because$ pole!

(d) Pole of general order: If f has a pole of order n, then $f(z)=(z-z_0)^{-n}g(z)$ for $g: B_\delta(z_0)\longrightarrow \mathbb{C}$, $g(z_0)\neq 0$,

Hence...

$$res(f,z_0) = \frac{1}{2\pi i}\int_{\partial B_\delta(z_0)} \frac{g(z)}{(z-z_0)^n}dz \overset{30}{=} \frac{g^{(n-1)}(z_0)}{(n-1)!}.$$

rewrite for f... pole w/ general order

so $$\boxed{res(f,z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right)}$$

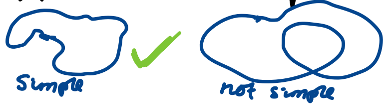
Corollary of Cauchy's integral formula
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw \quad (30)$$

know $f^{(n)}(z_0) = a_n n!$ & use formula for Taylor coefficients: $a_k = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz$

(e) General case: For essential singularities / have Laurent series, $\boxed{Res(f,z_0) = a_{-1}}$ Laurent series

Note: can now compute cool integrals, show $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ etc... see working / Q sheets for examples

58 Thm 9.18: (Argument principal). Let $\Omega\subset\mathbb{C}$ be a domain, $f:\Omega\longrightarrow\mathbb{C}_\infty$ be meromorphic, $f\not\equiv 0$.
If $P\subset\Omega$ are the poles & $Z\subset\Omega$ are the zeros, let $\gamma:[a,b]\longrightarrow\Omega\setminus(P\cup Z)$ be pw C^1 simple closed curve that bounds an open set $A\subset\Omega$ in the positive direction.
Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = Z_A(f) - P_A(f)$$



retraction $\gamma|_{[a,b)}$ fails injectivity

Note: If f has a pole of order n at z, $ord(f,z)=-n$, if $ord(f,z)\geq 1$, zero & taylor

$$Z_A(f) - P_A(z) = \sum_{z\in(Z\cup P)\cap A} ord(f,z)$$

$\boxed{ord(f,a) = \inf\{n\in\mathbb{Z}: a_n\neq 0\}}$

+ve in the case of a zero, −ve in case of pole

Pf: $f(z) = (z-z_0)^n g(z)$, $\frac{f'(z)}{f(z)} = \dots = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)} \Rightarrow \frac{f'}{f}$ simple pole w/ res n at z_0
so $res\left(\frac{f'(z)}{f(z)}, z_0\right) = ord(f,z_0)$ apply residue to each pole ■

↳ Cor 9.19: $Z_A(f) - P_A(f) = \underbrace{I(f\circ\gamma, 0)}_{\text{winding \# of } f\circ\gamma}$ Pf: $\int_\gamma \frac{f'(z)}{f(z)}dz = \dots = \int_{f\circ\gamma}\frac{1}{z}dz$

59 Thm 9.21: Let $g, G:\Omega$ (domain) $\longrightarrow\mathbb{C}$ holo. $\gamma:[a,b]\longrightarrow\Omega$ pw C^1 simple closed curve that bounds open set $A\subset\Omega$ in a positive direction ($I(\gamma,z)=1$ $\forall z\in A$ & two connected components), suppose $|g(z)|<|G(z)|$ $\forall z\in\gamma([a,b])$. Then G and $G+g$ have the same # of zeros in A.

adding some noise of may move zeros / change multiplicity but # const.

Pf: Apply the dog walking lemma (20) with G & $G+g$ $\Rightarrow$ $\underbrace{I(G\circ\gamma,0)}_{\text{\# zeros}} \overset{58}{=} \underbrace{I((G+g)\circ\gamma,0)}_{\text{\# zeros}}$ ■

Note: Classic exam Q here. Pick simple function $G=\{$know where zeros are$\}$, $|g|<|G|$ &

Chapter 10 – Sequences of Holomorphic Functions

Real Anal: $\underbrace{x^n}_{\infty \text{ diff}} \longrightarrow \underbrace{|x|}_{\text{not diff}}$ on $\mathbb{R}$, no regularity!

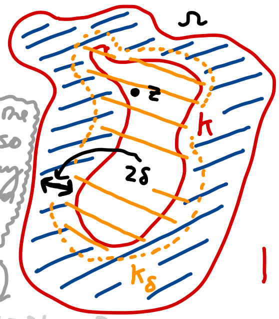
60 Thm 10.1: (Weierstrass convergence thm). $\Omega\subset\mathbb{C}$ open, $f_n:\Omega\longrightarrow\mathbb{C}$ sequence holo fntns on Ω.
If $f_n\longrightarrow f:\Omega\longrightarrow\mathbb{C}$ locally uniformly, Then

what you'd want it to be!

(i) f holomorphic (ii) $\forall k\in\mathbb{N}$, $f_n^{(k)}\longrightarrow f^{(k)}$ locally uniformly (higher derivatives)

Pf: $f_n \longrightarrow f$ uniformly, f_n cts $\overset{\text{AIII}}{\Longrightarrow}$ f continuous. **Morera's Thm:** cts & integrate around ∂Triangle, [31] then holo!

Goursat's thm: [24] $0 = \int_{\partial T} f_n(z)dz \xrightarrow{n\to\infty} \int_{\partial T} f(z)dz$ [uniform conv] $\Rightarrow f$ holo! (i) has been proved.



w lies on the boundary so further away from z than any pt in $B_\delta(z)$

$B_\delta(z) = \{\xi \in \mathbb{C} : |\xi - z| < \delta\}$
$\Rightarrow w \in \partial B_\delta(z)$ has $|w-z| > \delta$

For (ii), need actual analysis! Pick arbitrary compact $K \subset \Omega$. (WTS $f_n^{(k)} \longrightarrow f^{(k)}$ on this K)

Choose $\delta > 0$ s.t. $\forall z \in K$, $B_{2\delta}(z) \subset \Omega$, set $\underbrace{K_\delta = \bigcup_{z\in K} B_\delta(z)}_{\delta\text{ 'fattening' of } K}$ compact

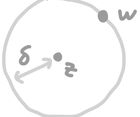
Think: "a little bigger"

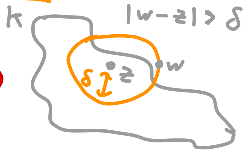
Formula from Taylor series — Think cauchy int. formula...

By (30), $f_n^{(k)}(z) - f^{(k)}(z) = \dfrac{k!}{2\pi i} \displaystyle\int_{\partial B_\delta(z)} \dfrac{f_n(w) - f(w)}{(w-z)^{k+1}} dw$ NOW CONTROL

$|w - z| > \delta$

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{2\pi} \sup_{w\in\partial B_\delta(z)} \left|\frac{f_n(w) - f(w)}{(w-z)^{k+1}}\right| \cdot \overbrace{\text{length}(\partial B_\delta(z))}^{=2\pi\delta}$$

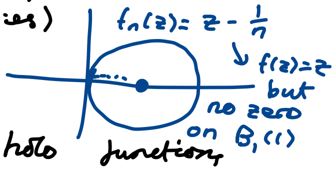




$$\leq \frac{k!}{\delta^k} \sup_{w\in K_\delta} |f_n(w) - f(w)| \xrightarrow[\text{of } f_n \longrightarrow f \text{ on } K_\delta]{\text{uniform converg}} 0$$ ■

(61) **Thm 10.2** (Hurwitz's thm): $\Omega \subset \mathbb{C}$ (open & connected), $f_n : \Omega \longrightarrow \mathbb{C}$ holo. $f_n \longrightarrow f$ locally uniformly. If $k \in \mathbb{N}_0$, then either (a) $f \equiv 0$ or (b) f has $\leq k$ zeros also (counting multiplicities)

↳ ie zeros can't jump up, but can jump ↓ (leave domain)



Pf: not covered in lectures so will ignore...

↳ **Cor 10.4:** Any $f : \Omega \longrightarrow \mathbb{C}$ arising as local uniform lmt of INJECTIVE holo functions $f_n : \Omega \longrightarrow \mathbb{C}$ (← open & connected) is either (a) constant (b) injective

Pf: Suppose f neither constant nor injective $\Rightarrow \exists z_1 \neq z_2$ s.t. $f(z_1) = f(z_2) = w$

Apply Hurwitz to $f_n - w \xrightarrow{\text{local uniform limit}} f - w$. $f_n(z) - w$ has at most 1 zero $\because f_n$ injective

$\because f$ not constant, $f - w \not\equiv 0 \Rightarrow$ by Hurwitz, $f(z) - w$ has at most 1 zero. ✗ $\because z_1$ & z_2 work ■

(62) **Def 10.5:** $\Omega \subset \mathbb{C}$ open. $f_n \longrightarrow f$ locally uniformly bbd if $\forall K \subset \Omega$ compact, $\exists M < \infty$ s.t. $|f_n(z)| < M$ $\forall z \in K$, $\forall n \in \mathbb{N}$

(63) **Def 10.6:** $K \subset \mathbb{C}$ compact. $f_n : K \longrightarrow \mathbb{C}$ uniformly equicontinuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall n \in \mathbb{N}$, $(\forall z, w \in K \text{ s.t. } |z - w| < \delta) \Longrightarrow |f_n(z) - f_n(w)| < \varepsilon$

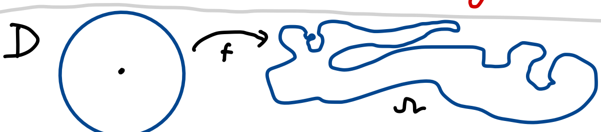
(64) **Thm 10.7:** (Ascoli - Arzelà). If $K \subset \mathbb{C}$ compact, $f_n : K \longrightarrow \mathbb{C}$ both uniformly bbd / equi-continuous. Then a subsequence converges uniformly to some $f : K \longrightarrow \mathbb{C}$ cts.

Pf: This is a special case of a Thm from Norms, Metrics & Topologies from year II. ■

(65) **Thm 10.8:** (Montel's thm). Every locally uniformly bbd sequence of holo $f_n : \Omega$ (open) $\longrightarrow \mathbb{C}$ has a locally uniformly convergent subsequence. ← will just recall main tool...

Pf: To estimate $|f(z_1) - f(z_2)|$ use cauchy integral formula! Stuff about diag subseq...

Chapter 11 – The Riemann mapping theorem



(66) **Thm 11.1:** Riemann mapping thm). Let $\Omega \subset \mathbb{C}$ be any simply connected domain other than $\mathbb{C}$. Then Ω is conformally equivalent to D (very, very general!)

↳ biholo f that always exists but cannot be written down...

(67) **Lemma 11.4:** Let Ω simply connected domain. $g : \Omega \longrightarrow \mathbb{C}\setminus\{0\}$ holo. Then $\exists$ holo $\ell : \Omega \longrightarrow \mathbb{C}$ s.t. $e^{\ell(z)} = g(z)$. In particular $\psi(z) = e^{\frac{1}{2}\ell(z)}$ is a holo sqrt of g.

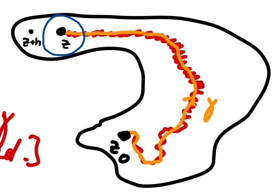
↳ e.g. take $g(z) = z \Rightarrow$ well defined log on Ω

Pf: By (38) $\Rightarrow$ IF $\exists$ holo F s.t. $F' = \frac{g'}{g}$ (this exists), then we can define $\ell(z) = F(z) + w_0$ & $g(z) = e^{\ell(z)}$ done!

↳ Need antiderivative $F(z)$ of $f(z) = \frac{g'(z)}{g(z)}$. Fix $z_0 \in \Omega$, $\forall z \in \Omega$ connect z to z_0 w/ piecewise C^1 curve γ. Set

$$F(z) = \int_{g\circ\gamma} \frac{dw}{w} = \int_\gamma \frac{g'(w)}{g(w)} dw$$ [Deformation thm $\Rightarrow \int$ independent of γ so well defined]

Also, F is holo & $F' = \frac{g'}{g}$



just append a line segment... $\qquad$ diff wrt h

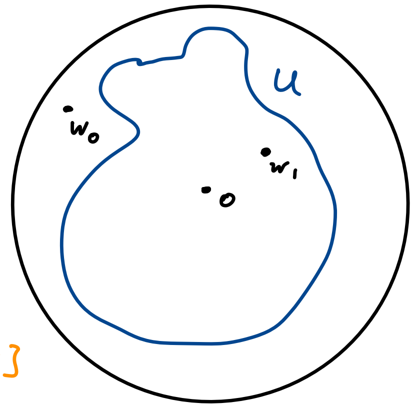
To see, set $F(z+h) = F(z) + \int_{[z,z+h]} \frac{g'(w)}{g(w)}\,dw$, $F'(z+h) = \frac{d}{dh}\int_{[z,z+h]} \frac{g'(w)}{g(w)}\,dw \Rightarrow \underbrace{F'(z) = \frac{g'(z)}{g(z)}}_{\text{as required}}$ ∎

Note: Schwartz lemma $\Rightarrow$ $H: D \longrightarrow D$ holo w/ $H(0)=0 \Rightarrow \underbrace{|H'(0)| < 1}_{\text{cannot stretch too much at origin}}$

shrinks the domain if there is room to stretch!

(68) Lemma 11.6: (Stretching lemma). $U \subset D$ simply connected. $0 \in U$ & $U \neq D$. Then, $\exists$ injective holo $H: U \longrightarrow D$ w/ $H(0)=0$ s.t. $|H'(0)| > 1$

$\frac{\frac{z-w}{\bar{w}z-1} - w}{\bar{w}\left(\frac{z-w}{\bar{w}z-1}\right)-1} = \frac{z - \cancel{w} - w(\bar{w}z - \cancel{1})}{\cancel{\bar{w}z} - \bar{w}w - \cancel{\bar{w}z} + 1} = \frac{z(1-w\bar{w})}{1-w\bar{w}} = z$



Pf: note $\forall w \in D$, $\phi_w(z) := \frac{z-w}{\bar{w}z-1}$ is biholo from $D \to D$ & it interchanges 0 & w, so $\phi_w \circ \phi_w = Id$ [own inverse]

Pick $w_0 \in D \setminus U$ & pick either w_1 s.t. $w_1^2 = w_0$ [square root]

Define $h: D \longrightarrow D$ w/ $h(0)=0$: $\phi_{w_0} \circ z^2 \circ \phi_{w_1} = h$

$h(z) = \phi_{w_0}([\phi_{w_1}(z)]^2)$. $z \longmapsto z^2$ not inj $\Rightarrow$ h not inj.

Schwartz lemma (42) $\Rightarrow |h'(0)| < 1$ [o/w, rotation − injective ✗.] (equality)

$h(0) = \phi_{w_0}(\underbrace{[\phi_{w_1}(0)]^2}_{\underbrace{w_1}_{w_0}}) = 0$

$\because 0 \notin \phi_{w_0}(U)$, use (67) $\exists \psi: U \longrightarrow \mathbb{C}$ holo 'square root' s.t. $\psi(z)^2 = \phi_{w_0}(z)$ $\forall z \in U$. $\psi: U \longrightarrow D$ even! why?

$\psi(0)^2 = (\phi_{w_0}(0))^2 = w_0 = w_1^2$. wma $\psi(0) = w_1$ [might need flip sign]

How def inj. holo $H: U \longrightarrow D$ w/ $H(0)=0$ via $H = \phi_{w_1} \circ \psi$. why?

$h \circ H(z) = \phi_{w_0}([\underbrace{\phi_{w_1} \circ \overbrace{\phi_{w_1} \circ \psi(z)}^{H(z)}}_{Id}]^2) = \phi_{w_0} \circ \phi_{w_0}(z) = z$ so diff w/ chain rule

$\frac{d}{dz}(h(H(z))) = h'(H(z)) \cdot H'(z) = 1$, at $z=0 \Rightarrow |H'(0)| = \frac{1}{|h'(H(0))|} = \frac{1}{|h'(0)|} > 1$ ∎

(69) Proof of Riemann mapping: will divide the proof into 3 separate claims.

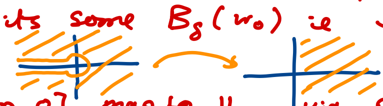
Bare hands construction here...

(a) Claim 11.7: The domain Ω is conformally equivalent to some open subset of D.

Pf: Claim trivial if Ω bdd: $\Omega \subset B_R(0) \Rightarrow \phi(z) = \frac{z}{R}$ gives conformal equivalence.

More generally, if Ω omits some $B_\delta(w_0)$ ie $\Omega \cap B_\delta(w_0) = \phi$, set $\phi(z) = \frac{\delta}{z - w_0}$

doesn't omit ANY open ball

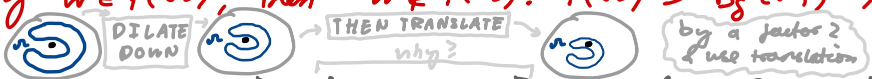


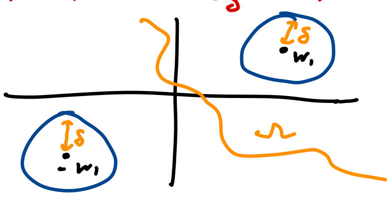
[makes some space!]

Remaining case is $\mathbb{C} \setminus (-\infty, 0]$. map to $H_{Re>0}$ via $re^{i\theta} \longmapsto \sqrt{r}e^{\frac{i\theta}{2}}$ for $\theta \in (-\pi, \pi)$

generalise this idea [Then, $\because \Omega \neq \mathbb{C}$, assume $0 \notin \Omega \Rightarrow \psi: \Omega \longrightarrow \mathbb{C}$ s.t. $\psi(z)^2 = z$ [image of Ω under ψ is $\frac{1}{2}\mathbb{C}$]

e.g., if $w \in \psi(\Omega)$, Then $-w \notin \psi(\Omega)$. $\psi(\Omega) \supset B_\delta(w_1) \Rightarrow \psi(\Omega)$ omits $B_\delta(-w_1)$ ∎





so (a) $\Rightarrow$ can assume $\Omega \subset D$. Assume $0 \in D$. Consider the set

$\mathcal{F} = \{f: \Omega \longrightarrow D : f \text{ holo, injective, } f(0) = 0\}$

Note $z \longmapsto z \in \mathcal{F}$ so $\mathcal{F} \neq \phi$. GOAL: show $\mathcal{F}$ contains a surjective function!

(b) Claim 11.8:

Sneaky Exam Tricks

ⓐ **Q1e 2021 Exam:** show $\displaystyle\int_0^{2\pi} \frac{1}{1+a\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{1-a^2}}$

Trick: If ∂D parametrised by $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = e^{i\theta}$, then $\gamma'(\theta) = ie^{i\theta} = i\gamma(\theta)$ so

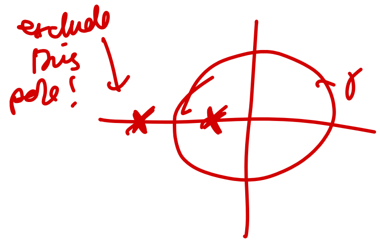
$$\int_\gamma \frac{f(z)}{iz}\,dz = \int_0^{2\pi} \frac{f(\gamma(\theta))}{i\gamma(\theta)}\gamma'(\theta)\,d\theta = \int_0^{2\pi} \underbrace{f(e^{i\theta})}_{\text{pick to be whatever you want!}}\,d\theta$$

So $\displaystyle f(z) = \frac{1}{1+a\left(\frac{z+z^{-1}}{2}\right)} = \frac{2z}{2z+az^2+a}$

Then

$$\int_0^{2\pi}\frac{1}{1+a\cos\theta}\,d\theta = \int_\gamma \frac{2}{i(az^2+2z+a)}\,dz$$

Then residue thm



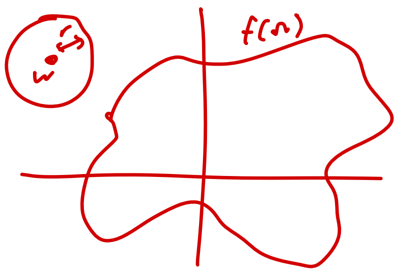
ⓑ **Q2a 2021 Exam:** $f : \mathbb{C} \to \mathbb{C}$ holomorphic & image of f not dense in $\mathbb{C}$. Prove that f is a constant function.

there is some ball that the image doesn't reach

Pf: If $f(\Omega)$ not dense $\Rightarrow \exists\, w \in \mathbb{C}$ & $r > 0$ s.t. $B_r(w) \cap f(\Omega) = \emptyset$

Then $g(z) = \dfrac{1}{f(z)-w}$ is holomorphic & bounded because $\forall z \in \mathbb{C}$, $|f(w)-w| > r$, then

$$|g(z)| = \frac{1}{|f(z)-w|} < \frac{1}{r}$$



Liouville's thm says bdd entire function constant.

$\Rightarrow g \equiv a \in \mathbb{C}$

$\Rightarrow f = \dfrac{1}{g} + w \equiv \dfrac{1}{a} + w \Rightarrow f$ constant ■

ⓒ **Q6i 2017 Exam:** Ω simply connected, f holo on Ω, is $f(\Omega)$ necessarily simply connected? Justify!

Example: No! Take $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ & $f(z) = e^{2\pi i z}$

Then $f(x+iy) = \underbrace{e^{-2\pi y}}_{\le 1}\underbrace{e^{2\pi i x}}_{|\cdot|=1}$ so $|f| \le 1$, $y \to \infty$, $e^{-2\pi y} \to 0$

so $f : \mathcal{H} \to B_1(0)\setminus\{0\}$ NOT simply connected!

ⓓ **Q2e 2015 Exam:** WTS f constant & knew it was entire. WTS bounded...

Trick: Use that K compact, f continuous $\Rightarrow f(K)$ bounded (Topology)

ⓔ **Q3b 2018 Exam:** Solving the quartic $z^4 - 18z + 1$ from using trick ⓐ

Trick: $z^4 - 18z^2 + 1 = 0 \iff (z^2-9)^2 - 80 = 0$

$\iff z^2 = 9 \pm 4\sqrt{5} = \underbrace{(2\pm\sqrt{5})^2}_{\text{write as a square!}} \Rightarrow z = \pm 2 \pm \sqrt{5}$

(f) Proof of 7.9: Suppose $\Omega \subset \mathbb{C}$ open & connected. $g: \Omega \to \mathbb{C}\setminus\{0\}$ holo, and $\exists$ holo $F: \Omega \to \mathbb{C}$ s.t. $F' = \frac{g'}{g}$. Then, $\exists w_0 \in \mathbb{C}$ s.t. $\ell(z) := F(z) + w_0$ and we have $g(z) = e^{\ell(z)}$ $(+2\pi i n \quad n \in \mathbb{Z})$

WTS $g(z) \cdot \frac{1}{e^{\log g(z)}} = 1$

$e^{\overbrace{w_0 + F(z_0)}^{\ell(z)}} = g(z)$ w_0 uniquely determined up to $2\pi i \mathbb{Z}$

Pf: Fix $z_0 \in \Omega$ & pick $w_0 \in \mathbb{C}$ s.t. $\underbrace{e^{w_0} = g(z_0) e^{-F(z_0)}}_{\text{achieve what we want everywhere at just 1 pt \& go out...}} \Rightarrow$ have $g(z) e^{-\ell(z)}$

$\underbrace{(g(z) e^{-\ell(z)})'}_{\text{say} = c} = g'(z) e^{-\ell(z)} - g(z) e^{-\ell(z)} \ell'(z) = e^{-\ell(z)} \underbrace{(g'(z) - g(z) F'(z))}_{F' = \frac{g'}{g}} = 0$

constant throughout. $\Rightarrow c = g(z_0) e^{-\ell(z_0)} = g(z_0) e^{-F(z_0)} e^{-w_0} = 1 \Rightarrow g(z) = e^{\ell(z)}$ ∎

Questions to review

- 2021 Exam: some of Q1, all of Q2 ☐
- 2017 Exam: Q1c, Q1d, Q2c, Q2d, Q4b, Q5b, Q5c ☐
- 2015 Exam: Q1fii, Q2d, Q2e, Q4 (Chapter 10), Q5 (log!) ☐
- (2018 Exam: Q3b, Q3c, Q4c) ☐
- Q8.7, Q8.6, Q8.4