MA3F1

Introduction to Topology

1895 – ?

2/10/23

Note: 4 assignments

Lecture 1

Def: A metric space X is given by $d: X \times X \to \mathbb{R}$

(i) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$

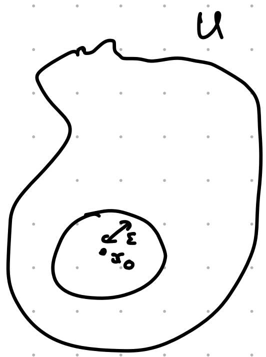
(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$

Example: $X = \mathbb{R}^n$, $d(x,y) = \|x-y\| = \left(\sum_i (x_i - y_i)^2 \right)^{\frac{1}{2}}$

Def: $B(x_0, \varepsilon) = \{ x \in X : d(x, x_0) < \varepsilon \}$ ← open ball

Def: $U \subset X$ open if $\forall x \in U$ $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subset U$



Note: ϕ & X are open, finite intersections of open sets open, arbitrary unions of open sets are open.

Def: A topology on a set X is a collection of sets $\mathcal{U}$ which satisfy

(i) $\phi \in \mathcal{U}$, $X \in \mathcal{U}$

(ii) If $u, v \in \mathcal{U} \Rightarrow u \cap v \in \mathcal{U}$ [finite intersections]

(iii) If $u_j \in \mathcal{U}$ for $j \in J \Rightarrow \bigcup_{j \in J} u_j \in \mathcal{U}$ [arbitrary unions]

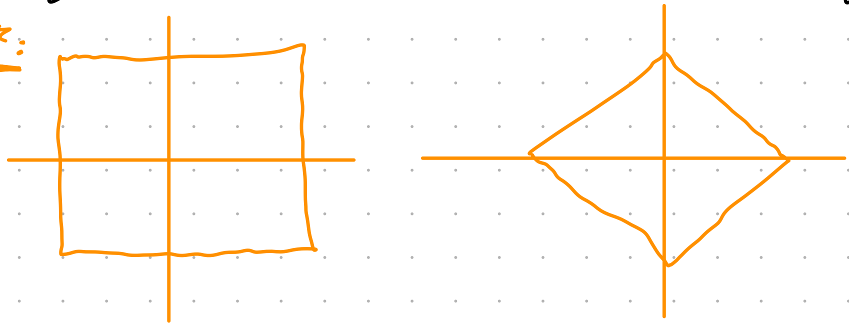
These are the open sets.

Continuous maps between topological spaces

Multiple metrics can produce the same topology...

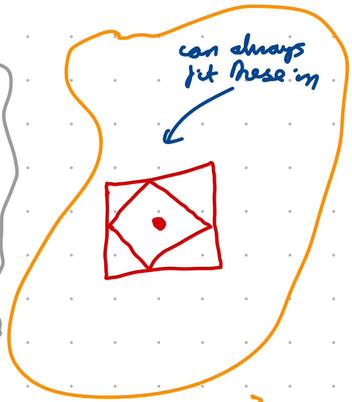
Example: Any norm $\|\cdot\|$ on $\mathbb{R}^n$ gives a metric $d(x,y) = \|x-y\|$ & all metrics on $\mathbb{R}^n$ coming from norms yield the same topology.

$\mathbb{R}^2$:



$\|\cdot\|_\infty = \max_i |x_i|$ $\qquad$ $\|\cdot\|_{L^1} = \sum_i |x_i|$

Different metrics but same topology ⇓ focus on topology, not metrics



$U \subset \mathbb{R}^2$

Shortcut to defining topologies:

Def: Say (X, Ω) is a topological space. A collection of sets $\mathcal{B} \subset \Omega$ is a basis for a topology if for each $U \in \Omega$ (each open set), there is a collection of open sets $\{B_j\}_{j \in J}$ in $\mathcal{B}$ s.t.

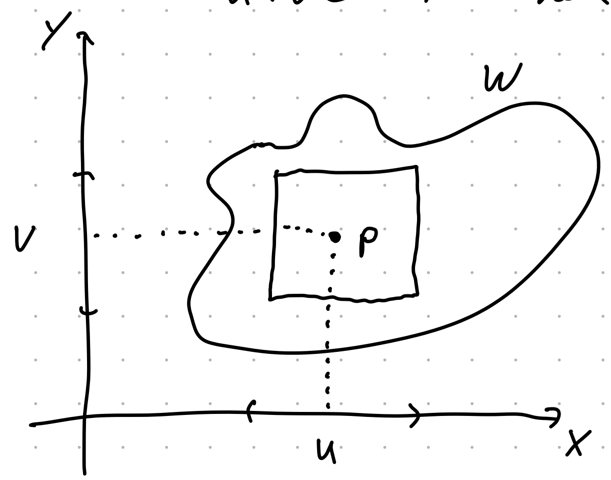
$$\bigcup_{j \in J} B_j = U$$

enough open sets to work out if an arbitrary set is open

Example: The open intervals (a,b) form a basis for the topology on $\mathbb{R}$.

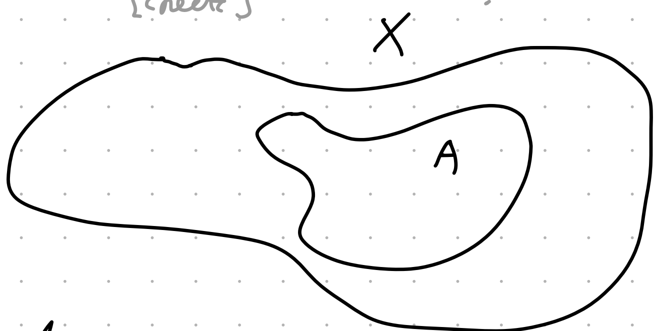
If $\mathcal{B}$ is a basis for the topology (X, Ω) we say that $\mathcal{B}$ generates the topology X. (a more efficient way to describe all the open sets, instead of just listing them).

Def: Let X & Y be topological spaces (Ω is implicity there). The product topology $X \times Y$ is the topology generated by sets of the form $U \times V \in X \times Y$ where U open in X, V open in Y.



W open if $\forall p \in W$, $\exists U$ open in X & $\exists V$ open in Y s.t. $U \times V$ open in X. ?

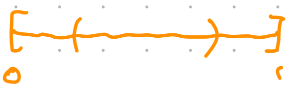
[check]



Def: Let A be a subset of a topological space X. The subspace topology on A corresponds to the collection of open sets $\Omega|_A$

$$\Omega|_A = U \cap A \quad \text{where } U \text{ is an open set for } X.$$

Example: Let $A = [0,1]$, $X = \mathbb{R}$, what is the subspace topology?



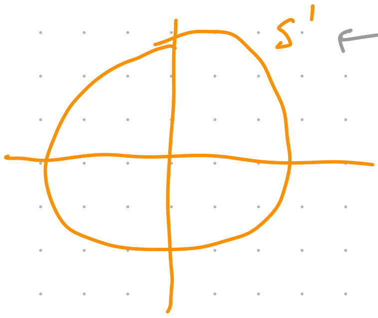
- The set $(\frac{1}{3}, \frac{2}{3})$ is open in A wrt the subspace topology (open in $\mathbb{R}$, & get set back when intersect w/ A)

- The set $(\frac{2}{3}, 1]$ is not open in X, but is open in A as

$$(\tfrac{1}{3}, 1] = [0,1] \cap \underbrace{(\tfrac{1}{3}, \tfrac{3}{2})}_{\text{open in } \mathbb{R}}$$

open wrt what is the key question

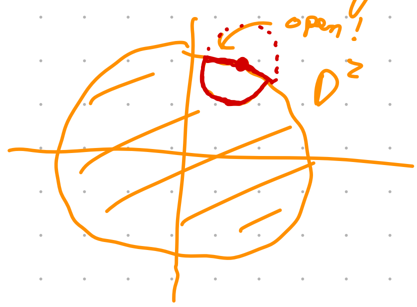
Example: The unit sphere. $S^n = \{x \in \mathbb{R}^{n+1} : \sum_i x_i^2 = 1\}$



$S^1 \leftarrow$ dimension of the circle, a 1d object.

S^n is a topological space wrt subspace topology

Disk $D^n = \{x \in \mathbb{R}^n : \sum_i x_i^2 \leq 1\}$



Def: Let X, Y be topological spaces. A ~~map~~ FUNCTION $f : X \longrightarrow Y$ is continuous if the inverse image of each open set in Y is an open set in X.
$\rightarrow$ compatible w/ metric spaces but more general...

Notation: $\boxed{\text{we call a cts function a map}}$ [for ease]

Example:
- $1_X : X \longrightarrow X$ is cts (identity map)
- The inclusion of A into X (where A is given the subspace topology) is cts $f : A \to X$.
- The function $t \longmapsto (\cos(t), \sin(t))$ from $\mathbb{R}$ to $\mathbb{R}^2$ is cts (Note, will often write this as $t \mapsto e^{it}$ where (x, y) is identified with $x + iy$)
- Compositions of cts functions are cts

Lemma: Let X be a topological space $X = A \cup B$ where A, B are closed subspaces of X. If $f : X \longrightarrow Y$ is a function & $f|_A$ & $f|_B$ are both cts, then f itself is continuous.
create new cts from old cts by pasting them together.

Def: $f : X \to Y$ is a homeomorphism if there is a map (is cts) $g : Y \to X$ s.t. $f \circ g = 1_Y$ & $g \circ f = 1_X$.

Trying to understand spaces upto the relation of being homeomorphic
what does it mean for two spaces to be from

Lecture 2 Topology is always upto homeomorphism **3/10/23**

Thm (Invariance of Domain, 1910): If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then $n = m$.
Easy to show there's no linear isomorphism (just linear algebra), homeomorphism is hard.

Exercise: $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$

$\mathbb{R}^2$ " " " " " for $n > 2$ [This course]

$\mathbb{R}^3$ " " " " " for $n > 3$ [Intro to algebraic topology]

Comment about subspace topology: Say X, Y are topological spaces. $f : Y \to X$ is a map taking values in $A \subset X$. Then there is a g with $f = \iota \circ g$. With respect to the subspace topology, g is continuous.

$$\begin{tikzcd} & X \\ Y \arrow[ur, "f"] \arrow[r, dotted, "g"'] & A \arrow[u, "\iota"'] \end{tikzcd}$$

inclusion map here

[Different ranges. One mapping to A, one mapping to X]

Proof: EXERCISE

Def: Let $\{X_j\}_{j \in J}$ be a family of topological spaces. The disjoint union of this family is the topological space w/ underlying set

$$\bigsqcup_{j \in J} X_j = \{(x_0, j) : x_0 \in X_j\}$$

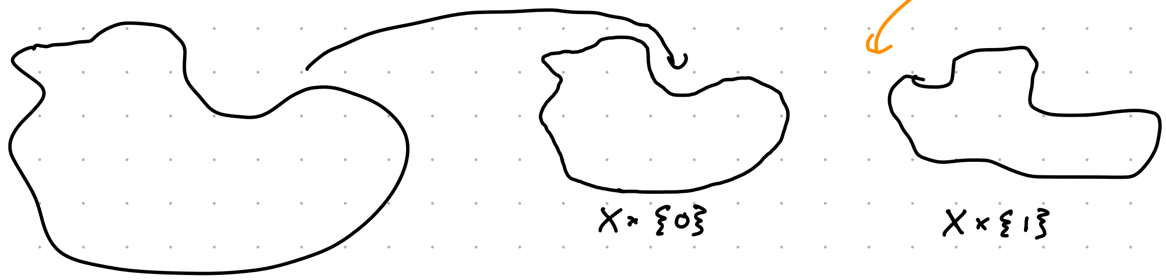
where the topology is generated by the basis of sets of the form $U \times \{j\}$ for $j \in J$ (indexing set) & U an open set in X_j.

Example Say $j \in \{0,1\}$. Fix a space X

$$X \sqcup X = (X \times \{0\}) \cup (X \times \{1\})$$

so

added extra coordinate



$X \times \{0\}$ $\quad$ $X \times \{1\}$

$X = X \cup X$ $\qquad$ $X \sqcup X$

two distinct copies rather than $X \cup X = X$

Quotient Spaces

Recall, an equivalence relation on a set X is a subset $E \subset X \times X$ s.t.

1. For $x \in X$, $(x,x) \in E$
2. If $(x,y) \in E$, then $(y,x) \in E$
3. If $(x,y), (y,z) \in E$, then $(x,z) \in E$

Usually write $x \sim y$, not $(x,y) \in E$. Equivalence relations partition X into equivalence classes. Write $[x] = \{y : x \sim y\}$

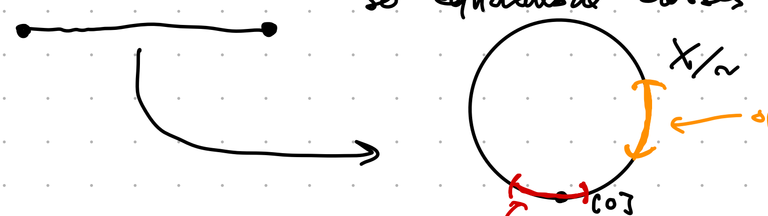
The set of equivalence classes is written $X/\sim$. The map $q : X \to X/\sim$ written $q(x) = [x]$ is called the quotient map.

How interact w/ topology?

Def: Let X be a top. space w/ an equivalence relation on the underlying set X. The quotient topology $X/\sim$ has open sets, those $V \in X/\sim$ for which $q^{-1}(V) = \{x \in X : q(x) \in V\}$ is open in X.

Example: $X = [0,1]$. $0 \sim 1$ and $x \sim x$ $\forall x$

so equivalence classes are $\{0,1\}$ & $\{x\}_{x \in (0,1)}$



X $\qquad$ $X/\sim$

Which sets in $X/\sim$ are open?

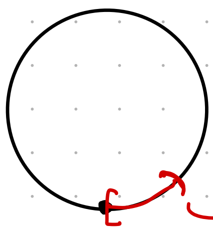
open interval, inverse image open so open.

$[0]$

the inverse image is

two open intervals
open in X. so open.

What about



inverse image

contains this singleton :: of the equivalence relation

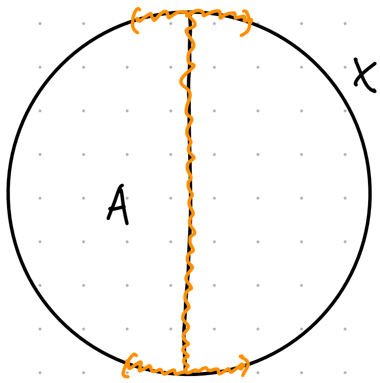
we get the topology we expect from the circle by doing this construction w/ the equivalence relation on a line!

Getting the topology we expect!

<u>Example:</u> Can generalise the above. Say $A \subset X$, define an equivalence relation so that any two pts equivalent. $a_0 \sim a_1$ for $a_0, a_1 \in A$ and $x \sim x$ for any $x \in I$. Equivalence classes here are singletons & the set A.

$X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} \cup \{(0,y) : -1 \le y \le 1\}$

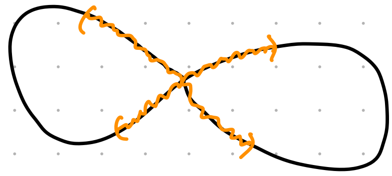
X is a unit circle, A is a segment of the circle



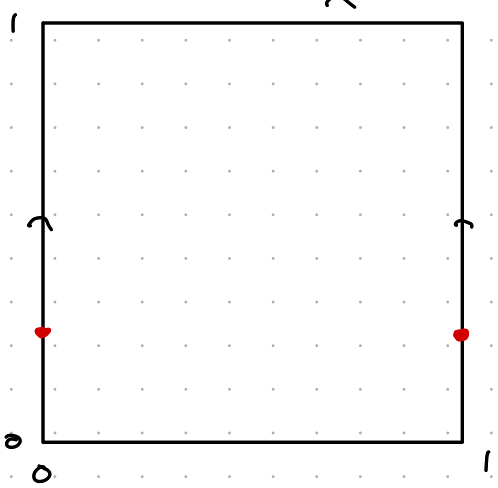
$\xrightarrow{q}$

quotient map

$X/\sim$



<u>Example:</u> $X = \{(x,y) : 0 \le x, y \le 1\}$



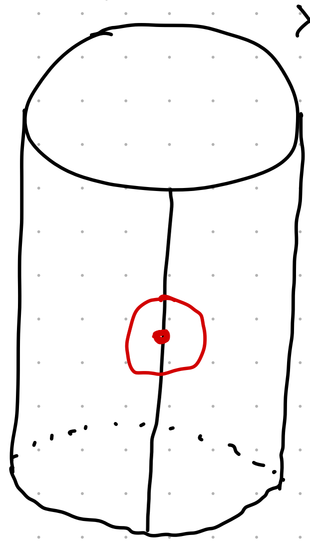
$(x,y) \sim (x,y)$ ← equivalence class, cardinality 1

$(0,y) \sim (1,y)$ ← equivalence class, cardinality 2

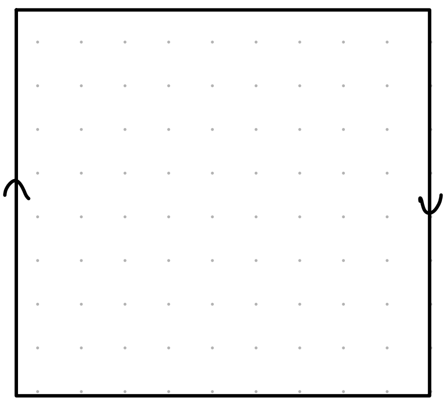
q

cylinder

$X/\sim$



Mobius strip

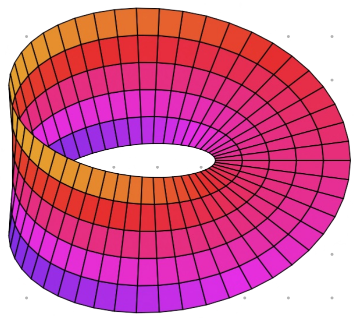


X

$(x,y) \sim (x,y)$

$(0,y) \sim (1,1-y)$

q

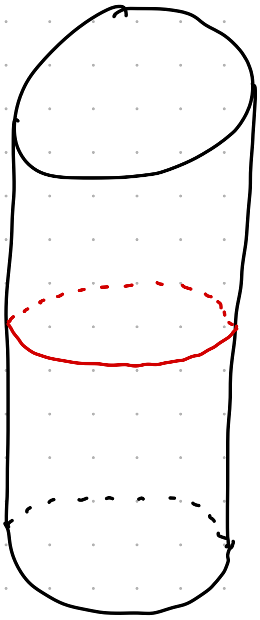


$X/\sim$

Lecture 3 3/10/23

$X \cong Y$ homeomorphic is one interesting relation in topology. There is another looser one (homeotopy). weaker. Will build up to the definition of this weaker relationship than homeomorphism.

Example:



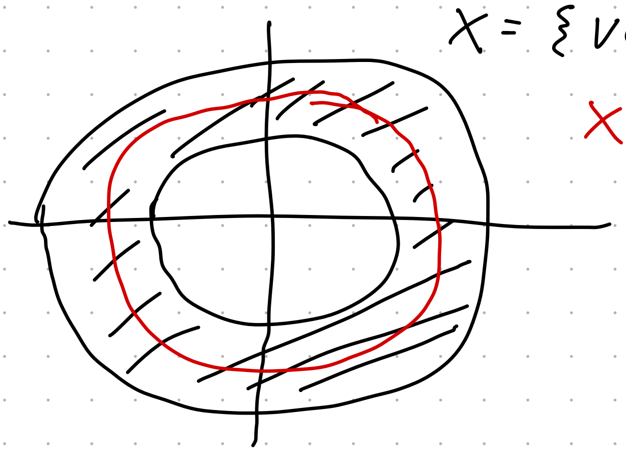
Let X be the cylinder.

Let A be the circle inside the cylinder.

Instead, lets switch models

$X = \{ v \in \mathbb{R}^2 : \frac{1}{2} \leq \|v\| \leq \frac{3}{2} \}$

$X = \{ v \in \mathbb{R}^2 : \|v\| = 1 \}$



Def: we say a topological space $A \subset X$ is a retract of X if there is a map $r: X \longrightarrow A$ with $r|_A = Id_A$

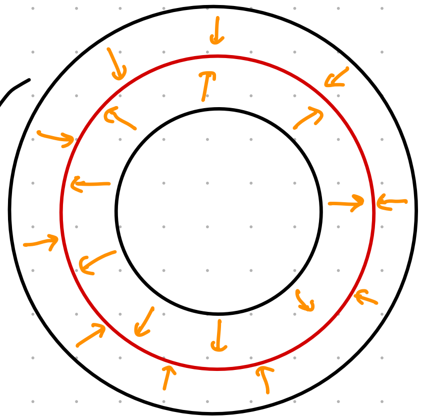
Example: There is a retraction $r: X \longrightarrow A$, $r(v) = \frac{v}{\|v\|}$ "same shape"

you can get your hands on this

Def: $A \subset X$ is a deformation retract of X if there exists a parameter family of functions $f_t : X \longrightarrow X$, $t \in [0,1] = I$ such that

$$f_0 = Id_X, \quad f_1(X) = A, \quad f_t|_A = Id_A$$

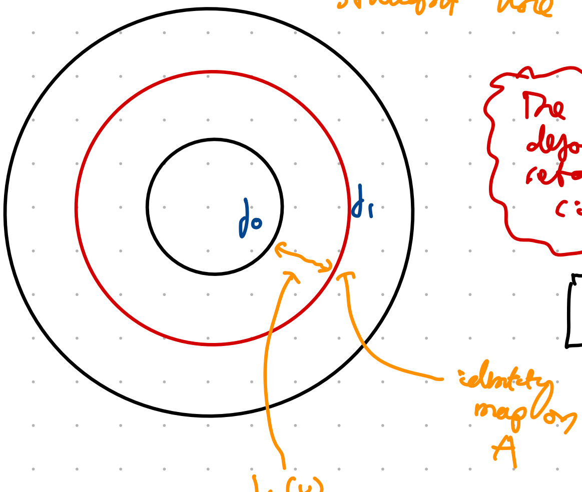
& the map $X \times I \longrightarrow X$, $(x,t) \longmapsto f_t(x)$ is cts



Example: $f_t(v) = (1-t)v + t\frac{v}{\|v\|}$

straight line

$f_t(x) = F(x,t)$ where ...



The circle is a deformation retract of the circle

$\boxed{f_1 = \iota \circ r}$

The map

$$\begin{tikzcd} X \arrow[rr, "f_1"] \arrow[dr, "r"'] & & X \\ & A \arrow[ur, "\iota"'] & \end{tikzcd}$$

Define $r(x) = f_1(x)$

$r: X \longrightarrow A$

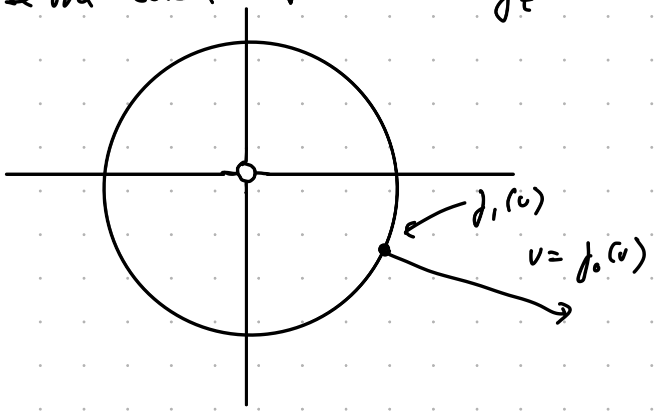
r is cts wrt subspace topology on X

$\longrightarrow$ conclude, $r: X \longrightarrow A$ is a retraction.

"circle & annulus have something in common. Deeper than homeomorphism" = retraction.

Example: S^{n-1} is a deformation retract of $\mathbb{R}^n/\{0\}$

use the same formula: $f_t(v) = (1-t)v + t\frac{v}{\|v\|}$



Will use these families of maps to define what a homotopy is

Example $\{pt\}$, say $\{0\}$ is a deformation contraction of $\mathbb{R}^n$

Pf: give f_t. $f_t(v) = (1-t)v$

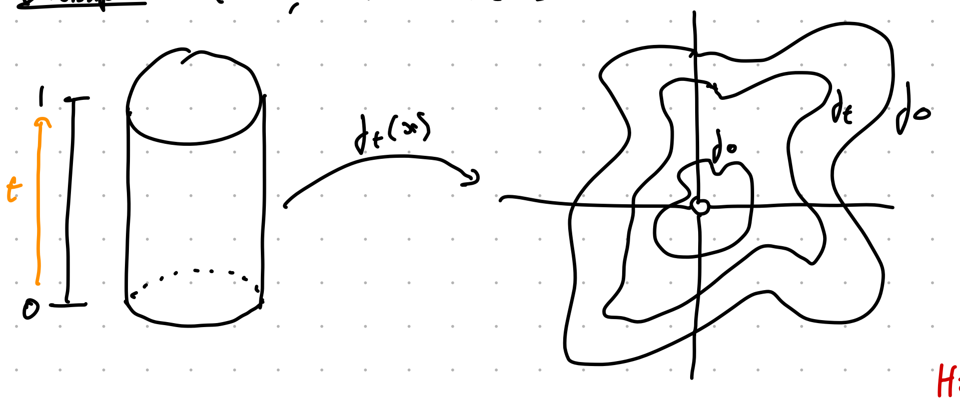
Homotopy

family of maps $f_t(x) = F(x,t)$

$I = [0,1]$

Def: Let X & Y be topological spaces. A map $F: X \times I \longrightarrow Y$ is called a homotopy. If $F(x,t) = f_t(x)$, then F is a homotopy from f_0 to f_1. Two maps f, g are homotopic if there exists a homotopy F s.t. $f_0 = f$ & $f_1 = g$.

Example: $X = S^1$, $Y = \mathbb{R}^2/\{0\}$



Continuous deformation

we say the maps are homotopic... $\downarrow$ next topic!

Higher level of abstraction...

Definition: Let X & Y be topological spaces. We say that X is homotopy equivalent to Y if there are maps $g: X \longrightarrow Y$ and $h: Y \longrightarrow X$ s.t.

$$\begin{tikzcd} X \arrow[r, bend left, "g"] & Y \arrow[l, bend left, "h"] \end{tikzcd}$$

family of maps

① $h \circ g \simeq Id_X$

② $g \circ h \simeq Id_Y$

not equals, but a homotopy (family of maps) exists.

We write $f \simeq g$ for homotopic.

we are suppressing the indices from above

general

Proposition: If A is a deformation retract of X, then A & X are homotopy equivalent.

Proof:

$$\begin{tikzcd} A \arrow[r, bend left, "\iota"] & X \arrow[l, bend left, "r"] \end{tikzcd}$$

Consider the inclusion ι & the retraction r

Show that $r \circ \iota = Id_A$, $\iota \circ r \simeq Id_X$

$\iota \circ r = f_0 \simeq Id_X$

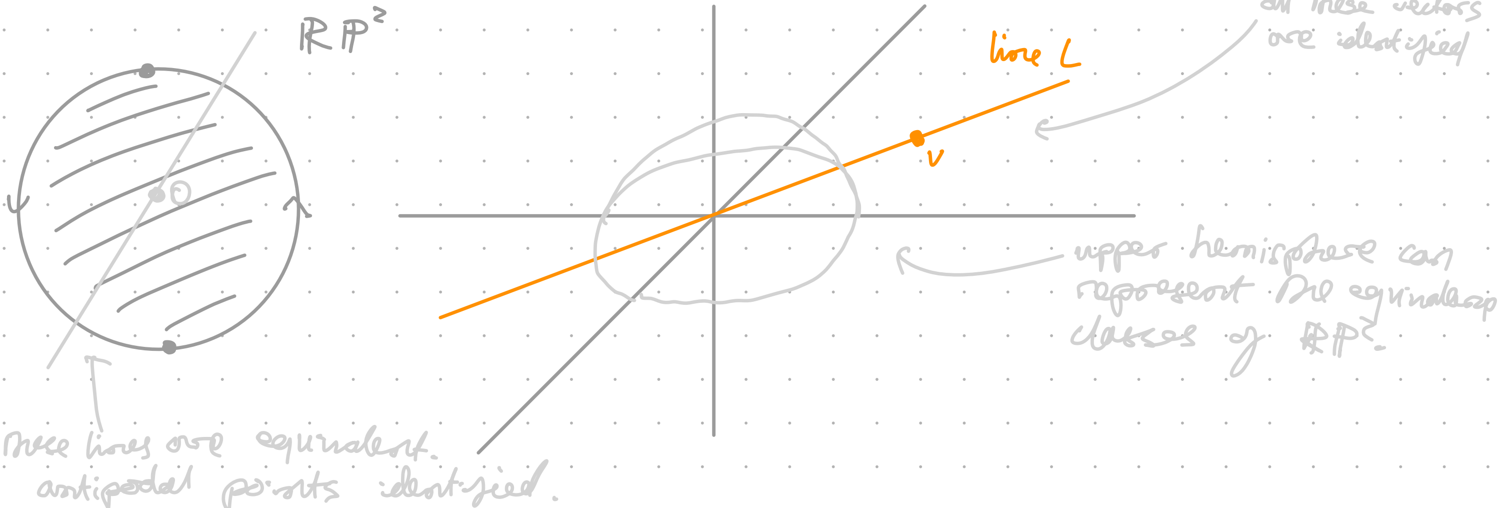
Lecture 4

9/10/23

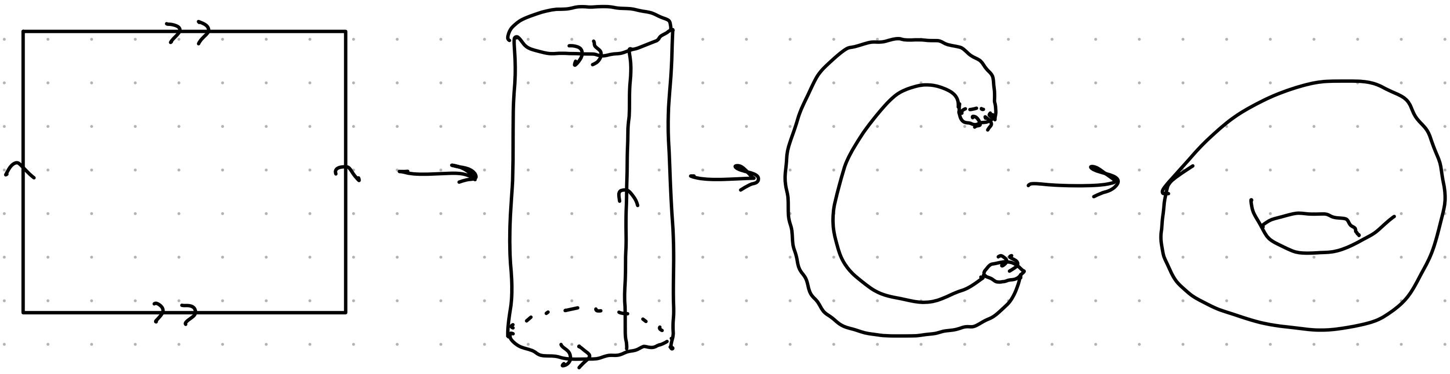
Examples of Topological spaces

① $\underbrace{\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\} \big/ v \sim rv \text{ for } r \in \mathbb{R}\setminus\{0\}}_{\text{real projective space}}$

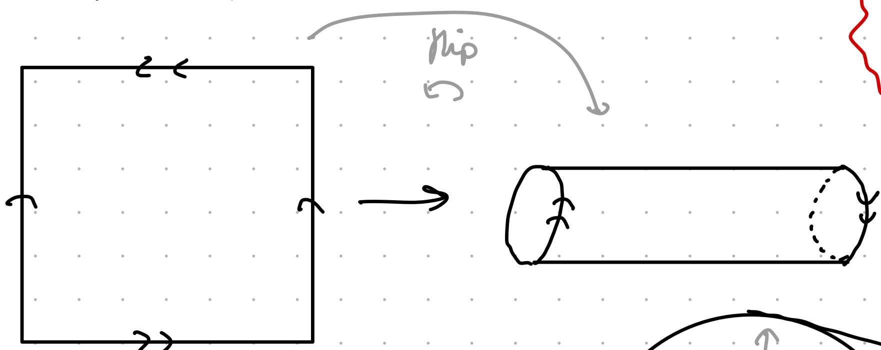
"$\mathbb{RP}^n$" is the space of lines in $\mathbb{R}^{n+1}$ =



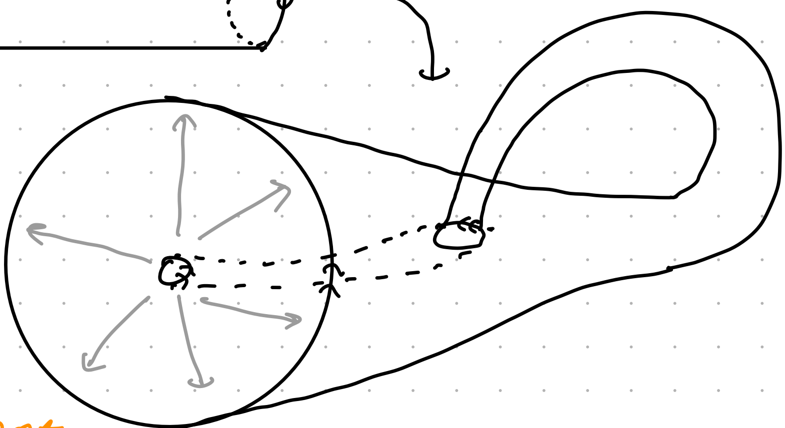
② Torus



③ Kline Bottle



All using the quotient topology



want some invariant to distinguish between homotopy equivalent spaces

Task: Find a homotopy invariant property of a topological space X. This will be a group we can attach to a space. We will consider loops in X where a loop has form

$f : [0,1] \longrightarrow X$ with $f(0) = f(1)$

will describe algebraic structure of the group & will calculate examples next week...

easier to define maps than to show they don't exist

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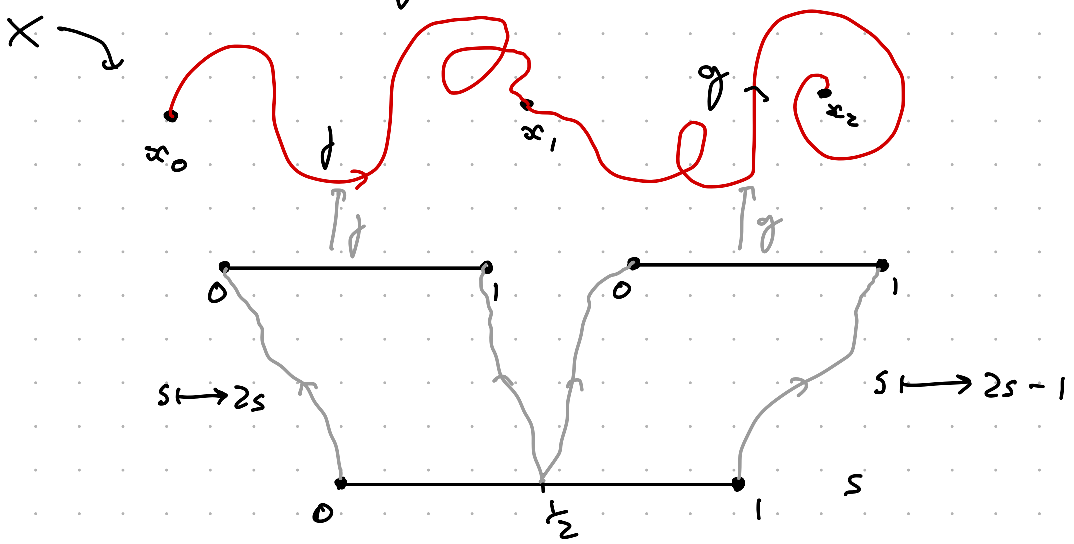
we want an operation on paths (loops will be a special case)

Concatenation

Say $f, g : I \longrightarrow X$ are paths (cts from $[0,1] \to X$) with special property that $f(1) = g(0)$. Want to define the concatenation $f \cdot g$ of f and g. In pictures:



definition of concatenation

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Note: $f \cdot g : [0,1] \to X$ is continuous by the pasting lemma.

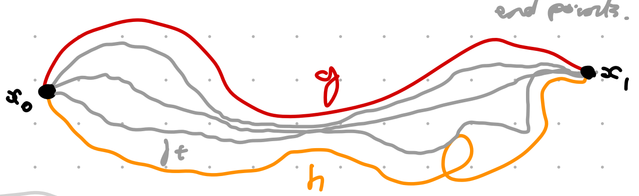
we're interested in homotopy classes of paths & loops.

homotopy for paths

For points $x_0, x_1 \in X$, consider paths $g, h : I \longrightarrow X$ satisfying $g(0) = h(0) = x_0$ and $g(1) = h(1) = x_1$. We say g & h are homotopic rel. endpoints if there is a homotopy $f_t : [0,1] \longrightarrow X$ with $f_0 = g$, $f_1 = h$ and $f_t(0) = x_0$ and $f_t(1) = x_1$ $\forall t \in [0,1]$

relative to the end points.

family of maps.

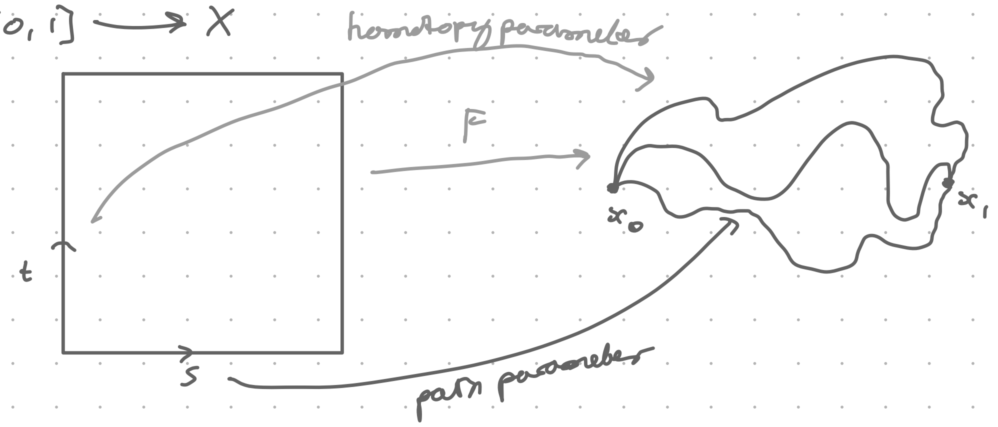


Paths are moving relative to fixed endpoints.

Lemma: Given $x_0, x_1 \in X$, the relation of homotopy rel. endpoints is an equivalence relation on the set of maps $f: [0,1] \to X$ w/ $f(0) = x_0$, $f(1) = x_1$

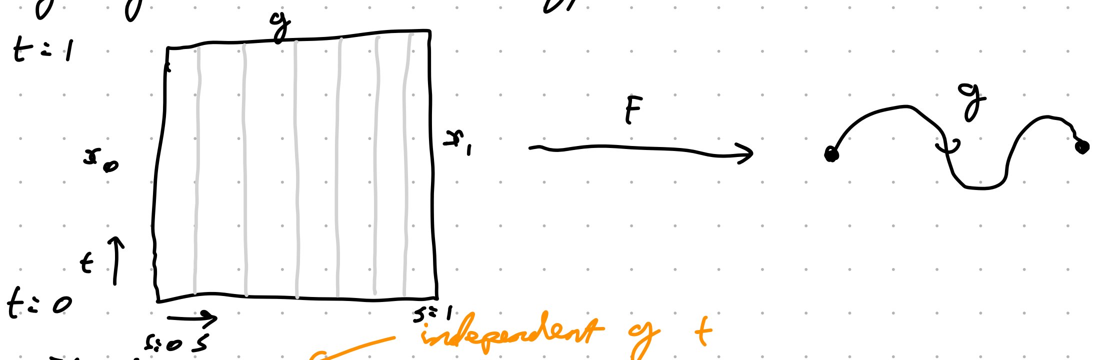
Remark: In the def, $f_t(s)$ is cts in both variables, hence

$F: [0,1] \times [0,1] \to X$



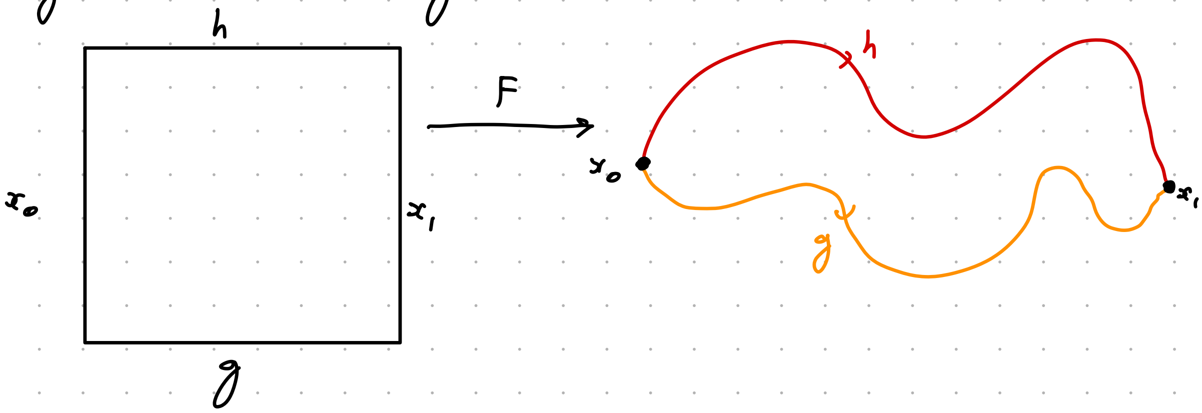
Proof: Need to show 3 things:

① $g \simeq g$ rel ∂ (boundary)

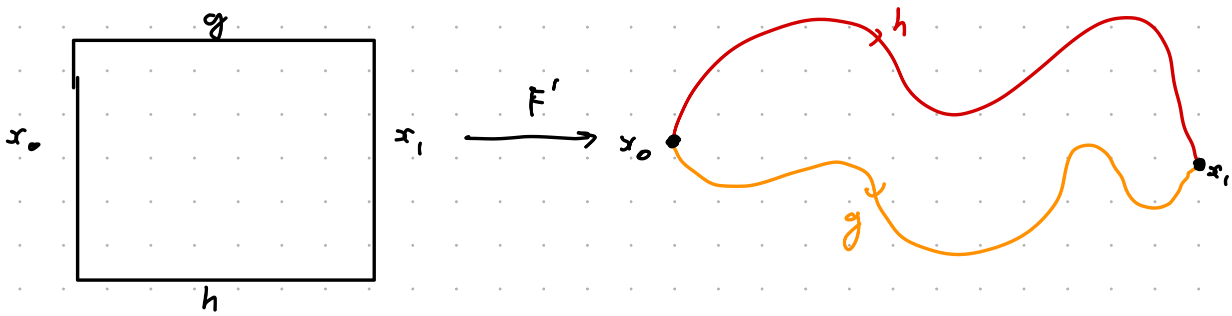


Let $F(s,t) = g(s)$ ← independent of t

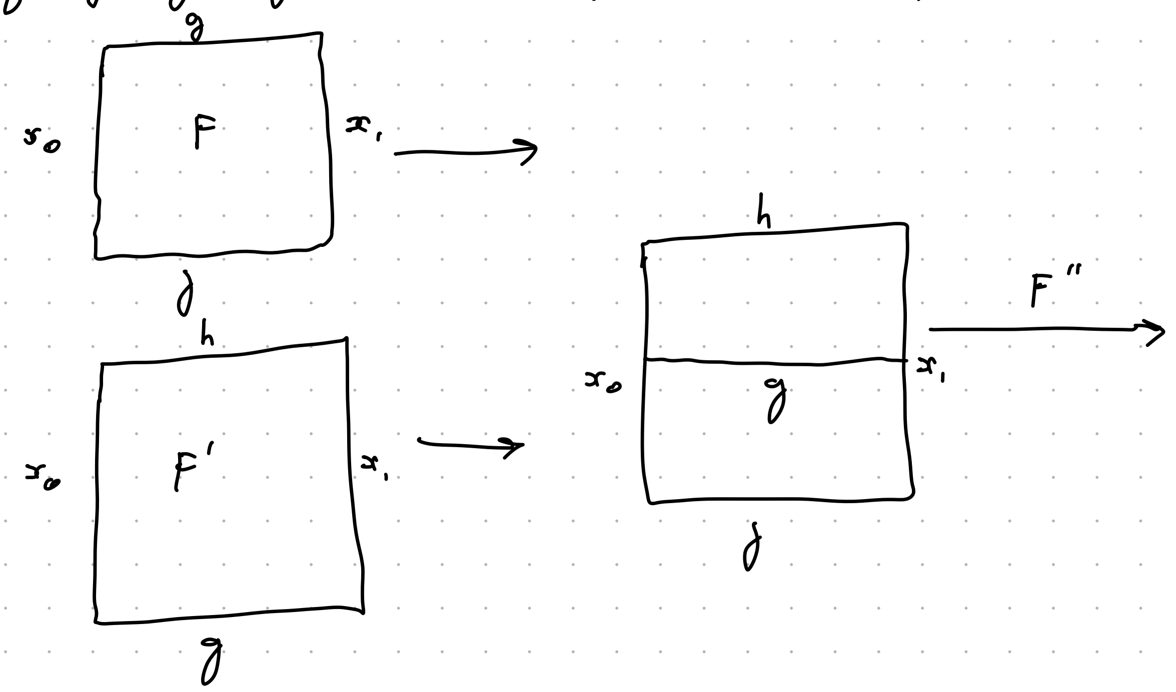
② If $g \simeq h$ then $h \simeq g$



Define $F'(s,t) = F(s, 1-t)$



③ Say $f \simeq g$, $g \simeq h$ then $f \simeq h$ (rel. ∂)



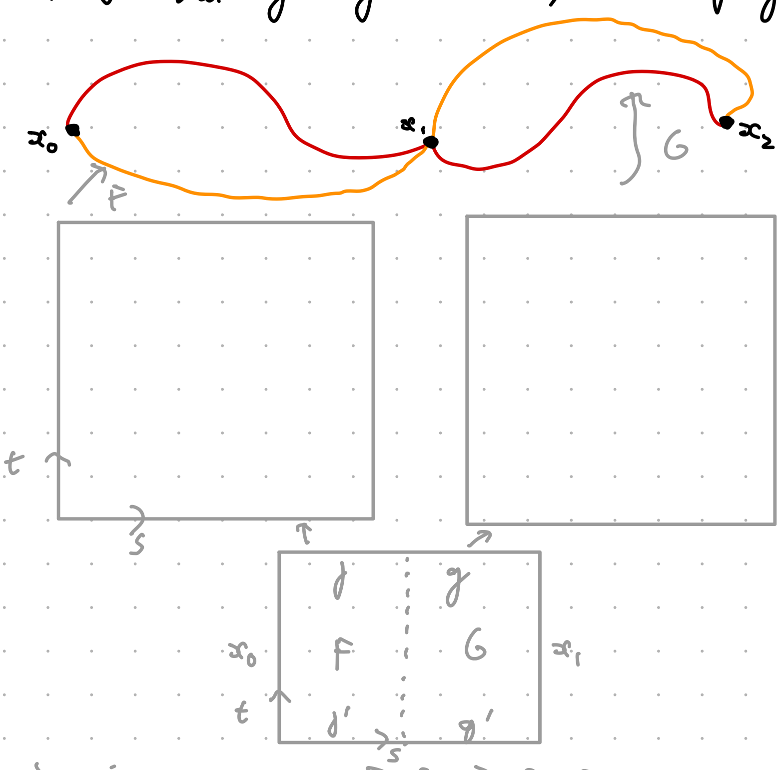
$$F''(s,t) = \begin{cases} F(s,2t) \\ F'(s,2t-1) \end{cases}$$

■

Homotopy rel ∂ is an equivalence relation on paths. 10/10/23

<u>Lemma</u>: Say f, f' are paths with $f(0) = f'(0) = x_0$, $f(1) = f'(1) = x_1$, g, g' paths with $g(0) = g'(0) = x_1$, $g(1) = g'(1) = x_2$. If $f \simeq f'$ rel ∂ and $g \simeq g'$ rel. ∂, then $f \cdot g \simeq f' \cdot g'$ rel ∂

<u>Proof</u>:



Not composition of functions, this is path concatenation so works left to right.

$f \simeq f'$ rel ∂ gives a map $F: [0,1] \times [0,1] \to X$

$g \simeq g'$ rel ∂ gives a map $G: I \times I \longrightarrow X$

Define
$$H(s,t) = \begin{cases} F(2s,t) & 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

H is cts by the pasting lemma

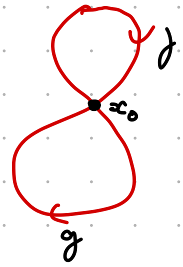
can think of concatenation as an operation on homotopy classes

If we have paths f, g with $f(1) = g(0)$, then we can define the operation of concatenation of homotopy classes by setting

$$[f]\cdot[g] = [f\cdot g]$$

[h] means a homotopy class rel ∂

well defined as a homotopy class



Def: Let X be a topological space. $x_0 \in X$, let $\pi(X, x_0)$ denote the set of homotopy classes of loops based at x_0. ie paths f with $f(0) = f(1) = x_0$. The operation of concatenation gives a "multiplication" on $\pi(X, x_0)$.

Def: $\pi_1(X, x_0)$ is the fundamental group of X.

Proposition: $\pi_1(X, x_0)$ is a group wrt concatenation

other cool stuff for π_2, π_3 etc..

Note: all loops are arbitrary, they can cross, hit x_0 again etc...

Think of π_1 as maps of circles into our space. π_n is spheres, hi dim analogue.

Proof: we need to show the existence of an identity, inverses & associativity.

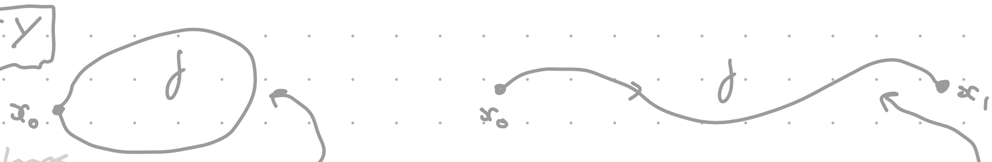
giving us a nice algebraic structure on this set — homotopy is good!

Observations:
1. All of these require the consideration of homotopy classes.
2. Each of these properties has a version for paths (talking about loops atm), in terms of the proof, we will prove the case of the path & apply to loops. Loops ⊂ paths so applies. will use these properties for paths to analyse rel. between fundamental group at different pts, $\pi_1(X, x_0)$ & $\pi_1(X, x_1)$

1. Let $e_0 : [0,1] \longrightarrow X$ be the constant path, $e_0(s) = x_0$

IDENTITY

It does exist!



loops

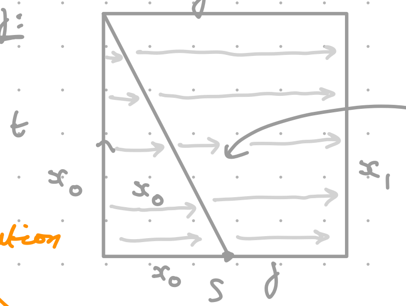
Claim: $e_0 \cdot f = f$ [e_0 is a left identity for f]

$f \cdot e_1 = f$ [e_1 is a right identity for f]

path

topologists don't like writing maps!

Pf:



$t = 0$ gives $e_0 \cdot f$

$t = 1$ gives f

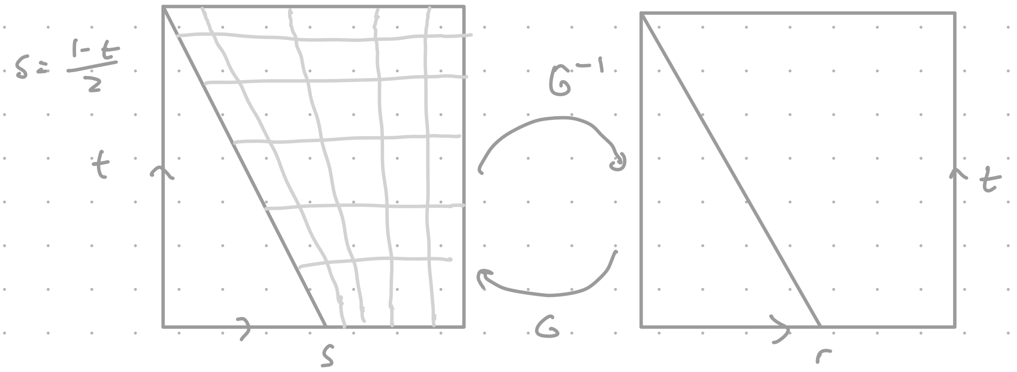
this is $s = \frac{1-t}{2}$ traverses f for longer & longer

$t = 0$, $s = x_0$

concatenation

$\underbrace{\quad}_{\text{constant}}$ $\underbrace{\quad}_{f}$

will write map here for novelty's sake...



required to do homotopies $\because$ the boundary ∂ is fixed

$s = (1-r)\frac{1-t}{2} + r$

$$G(r,t) = \left((1-r)\left(\frac{1-t}{2}\right) + r\cdot 1,\ t\right)$$

$r=0$, $r=1$ to parametrise the line

To find G^{-1}, we write $G(r,t) = (s,t)$ & solve for r as $r(s)$

$r(s) =$

$$F(s,t) = \gamma(\pi_r(G^{-1}(s,t)))$$

$$G^{-1}(s,t) = \left(\frac{2s+t-1}{t+1},\ t\right)$$

$s = \frac{t}{2}$

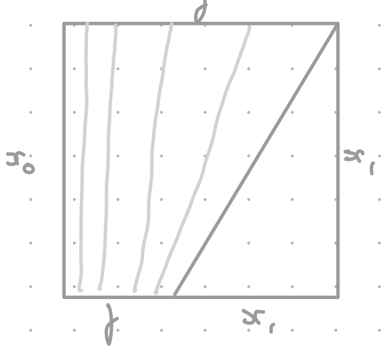
$\left(\frac{t}{t+1}, t\right)$

G^{-1} then project onto r plane then apply γ

So define homotopy F by

$$F(s,t) = \begin{cases} x_0, & 0 \le s \le \frac{1-t}{2} \\ \gamma\left(\frac{2s+t-1}{t+1}\right), & \frac{1-t}{2} \le s \le 1 \end{cases}$$

To show $\gamma \cdot e_1 = \gamma$ Do the same:



$$F(s,t) = \begin{cases} \gamma\left(\frac{2s}{1+t}\right) & 0 \le s \le \frac{t+1}{2} \\ x_1 & \frac{t+1}{2} \le s \le 1 \end{cases}$$

This fundamental group comes from the homotopy classes.

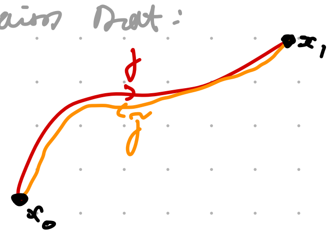
we've shown: $e_0 \cdot \gamma \simeq \gamma \simeq \gamma \cdot e_1$ rel ∂

↳ For the loop case, e_0 is a 2-sided identity

↳ for paths, we have a left identity & a right identity, not equal.

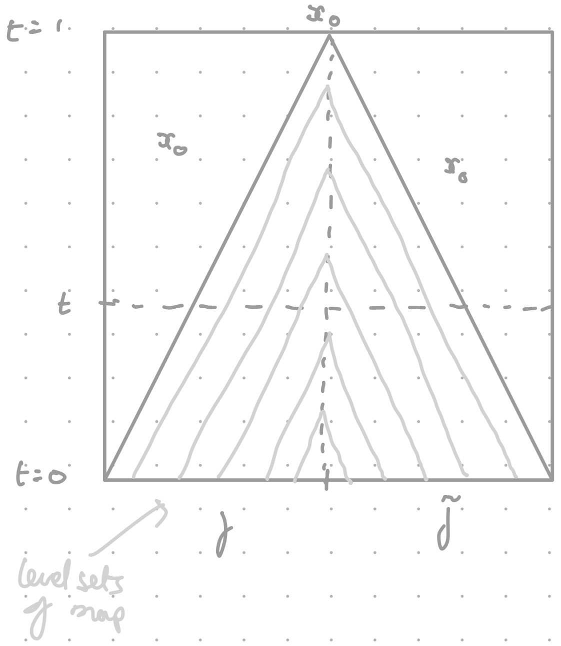
② (Inverses)

Let γ be a path from x_0 to x_1. Let $\tilde{\gamma}(s) = \gamma(1-s)$. Claim that:
$\gamma \cdot \tilde{\gamma} = e_0$, $\tilde{\gamma} \cdot \gamma = e_1$ [loop case, $e_1 = e_0$]



To show $[\gamma \cdot \tilde{\gamma}] = [e_0]$, we show $\gamma \cdot \tilde{\gamma} \simeq e_0$ rel ∂

So, we construct a map $F : I \times I \to X$



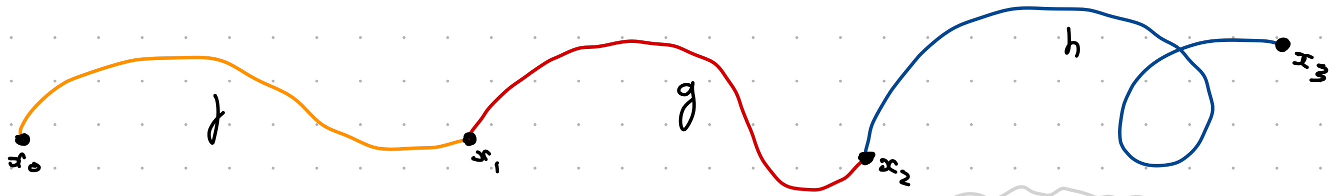
constant map

- t is our homotopy parameter
  ↳ fix t & we get a particular map
- Start at x_0, move along line, then go back to x_0 & wait longer till done.

$$F(s,t) = \begin{cases} x_0 & 0 \le s \le \frac{t}{2} \text{ or} \\ \gamma(2s-t) & \frac{t}{2} \le s \le \frac{1}{2} \\ \tilde{\gamma}(2s+t-1) & \frac{1}{2} \le s \le \frac{1-t}{2} \\ x_0 & 1-\frac{t}{2} \le s \le 1 \end{cases}$$

homotopy from $x_0 \longrightarrow x_1$ & all the way back vs just staying at x_0. + everything in between

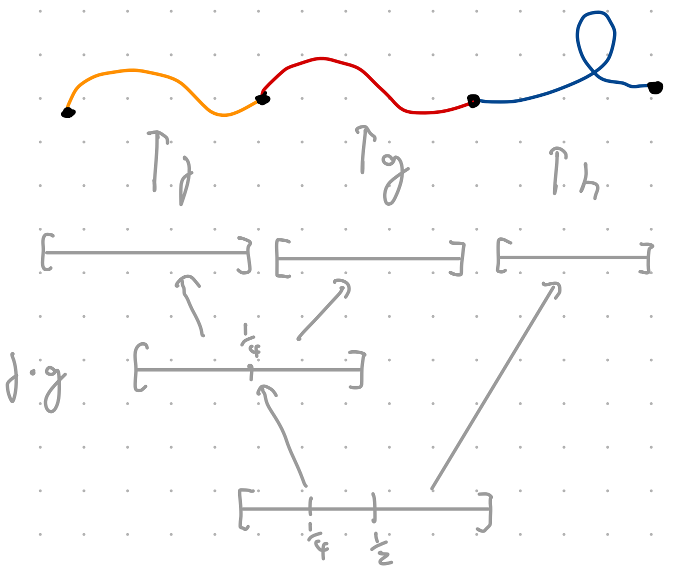
③ (Associativity) Assuming γ, g, h paths

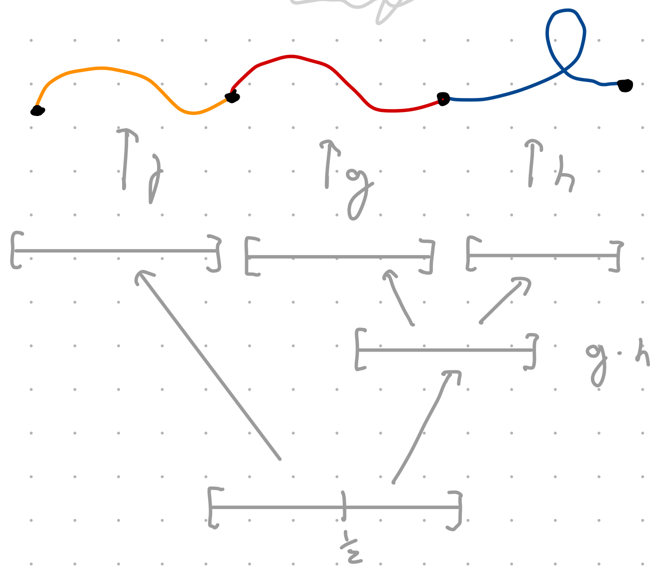


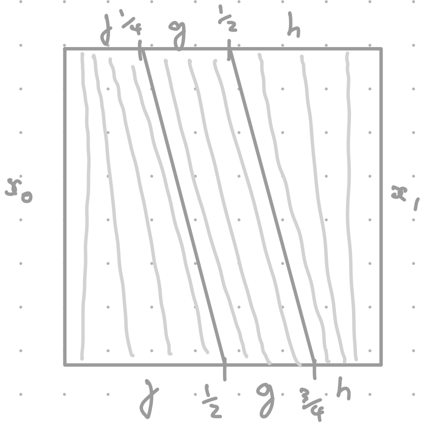
Claim: $[(\gamma \cdot g) \cdot h] = [\gamma \cdot (g \cdot h)]$

The difference is that they are parametrised differently

What's the difference?







$$F(s,t) = \begin{cases} \gamma\left(\frac{4s}{2-t}\right) & 0 \le s \le \frac{2-t}{4} \\ g(4s-2+t) & \frac{2-t}{4} \le s \le \frac{3-t}{4} \\ h\left(\frac{4s-3+t}{1+t}\right) & \frac{3-t}{4} \le s \le 1 \end{cases}$$

∎

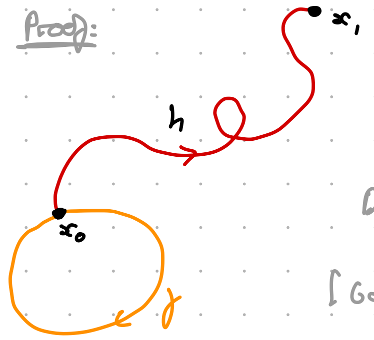
Building up more formal properties, we'll appreciate later...

Def: X is path connected if for any pair of points $x_0, x_1 \in X$, there is a path from x_0 to x_1. ← top. space.

Say $x_0, x_1 \in X$ and X is path connected. **Q:** What is the relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ fundamental group w/ different base points...

Prop: Say that $h : [0,1] \longrightarrow X$ is a path from x_0 to x_1, then h determines an isomorphism from $\pi_1(X, x_0)$ and $\pi(X, x_1)$. Once you're chosen h, iso class fixed.

Proof:



want $\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

want to move loops based at x_0 to loops based at x_1

Define $\beta_h([f]) = \bar{h} \cdot f \cdot h$ ← can write like this ∵ of associativity

[Go from $x_1 \longrightarrow x_0 \longrightarrow$ loop $\longrightarrow x_0 \longrightarrow x_1$] $[(\bar{h} \cdot f) \cdot h] = [\bar{h} \cdot (f \cdot h)]$

$\bar{h}$ goes back... β_h is a bijection from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$

claim that $\beta_{\bar{h}}$ is an inverse for β_h — use the fact that $\bar{\bar{h}} = h$

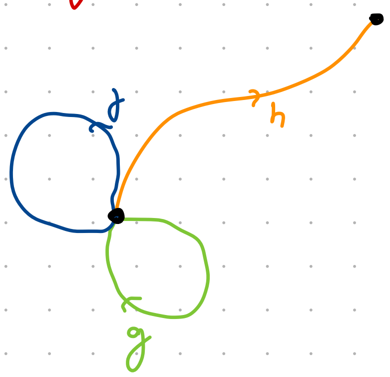
$$(\beta_{\bar{h}} \circ \beta_h)([f]) = [\bar{\bar{h}} \cdot (\bar{h} \cdot f \cdot h) \cdot \bar{h}]$$
$$= [(h \cdot \bar{h}) \cdot f \cdot (h \cdot \bar{h})]$$
$$= [e_0 \cdot f \cdot e_0]$$
$$= [f]$$

$(\beta_h \circ \beta_{\bar{h}})([f]) =$ same.

Shown:
$\beta_h \circ \beta_{\bar{h}} = Id$
$\beta_{\bar{h}} \circ \beta_h = Id$
$\Downarrow$
β_h bijection

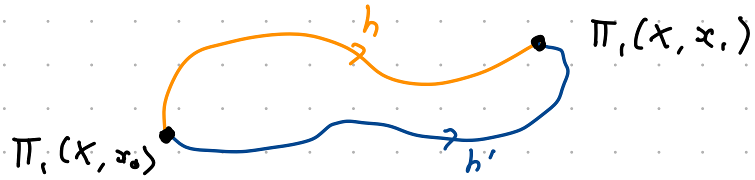
Claim: β_h is a group homomorphism

$$\beta_h([f] \cdot [g]) = [\bar{h} \cdot (f \cdot g) \cdot h]$$
$$= [\bar{h} \cdot f \cdot h \cdot \bar{h} \cdot g \cdot h]$$
$$= [\beta_h([f]) \cdot \beta_h([g])]$$



$\Rightarrow \beta_h$ is a group isomorphism

Does the isomorphism β_h depend on h? Answer: Yes in general it does



Happens if $\pi_1(X, x_0)$ is not abelian

If $\pi_1(X, x_0)$ is abelian, then the isomorphism is independent of the choice of path.

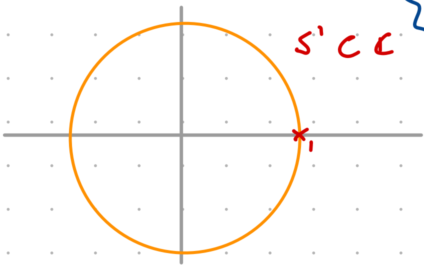
$\pi_1(S^1, x_0) = \mathbb{Z} \longrightarrow$ can ignore base point

$\pi_1(\text{Klein bottle}, x_0)$ is <u>not</u> abelian $\Rightarrow$ cannot ignore base point.

<u>Lecture ?</u> <u>16/10/23</u>

new subject so still learning how to teach it!

- Shown that $\pi_1(X, x_0)$ is a group



- WTS that $\pi_1(S^1, 1) = \mathbb{Z}$

  ↳ will take a while to prove...

- Will define a function $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$

  ↳ Given $n \in \mathbb{Z}$, define $\omega_n : I \longrightarrow S^1$ by $\omega_n(s) = e^{2\pi i n s}$

  ↳ This is a path! $\omega_n(0) = e^0 = 1$, $\omega_n(1) = e^{2\pi i n} = 1$

  ↳ This is a loop based at 1

  ↳ $\Phi(n) = [\omega_n] \in \pi_1(S^1, 1)$ ← homotopy class

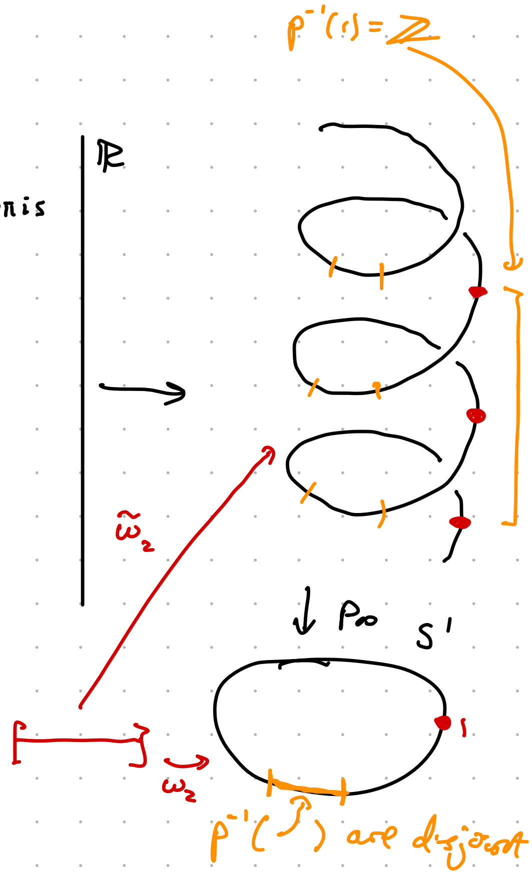
Fundamental groups & covering spaces interact

- Lets write $p_\infty : \mathbb{R} \longrightarrow S^1$ given by $p_\infty(s) = e^{2\pi i s}$

  The helix is the set of parameters

  $$(\cos(2\pi s), \sin(2\pi s), s) \subset \mathbb{R}^3$$

  $\rightarrow$ $p_\infty : \mathbb{R} \longrightarrow S^1$ is a covering space.



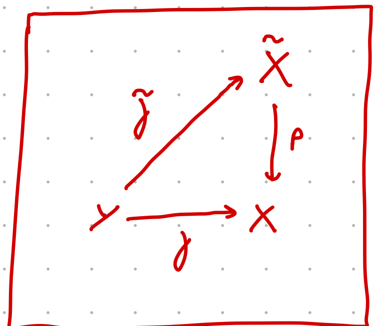
Def: Let $p : \tilde{X} \longrightarrow X$. A open set $U \subset X$ is evenly covered if $p^{-1}(U)$ is a disjoint union of open sets $\tilde{U}_j$ s.t. $p|_{\tilde{U}_j}$ is a homeomorphism.

← every pt

The map p is a covering map if X has a covering by evenly covered sets U.

The triple $p : \tilde{X} \longrightarrow X$ is a covering space

Def: Given a covering $p : \tilde{X} \longrightarrow X$ and a map $f : Y \longrightarrow X$, a lift of f is a map $\tilde{f}$ so that $p \circ \tilde{f} = f$



A lift diagram

<u>Example:</u> Define $\tilde{\omega}_n : [0,1] \longrightarrow \mathbb{R}$ by $\tilde{\omega}_n(s) = ns$. Then $p_\infty(\tilde{\omega}_n(s)) = e^{2\pi i n s} = \omega_n(s)$

Diagram:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p_\infty"] \\ {[0,1]} \arrow[ur, "\tilde{\omega}_n"] \arrow[r, "\omega_n"'] & S^1 \end{tikzcd}$$

Interpretation of $\tilde{\omega}_n(s_0)$ for $s_0 \in [0,1]$ is keeping track of the "total # of turns" of the path ω on $[0, s_0]$. ω is the actual point.

check later!

ω are loops
$\tilde{\omega}$ are paths] right way around...

1st Proposition

$$\boxed{\textbf{Prop: } \Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1) \text{ is a homomorphism}}$$

Proof: WTS $\Phi(n+m) = \Phi(n)\cdot\Phi(m)$ addition in $\mathbb{Z}$ ← composition of paths

↳ concretely, $\underbrace{[\omega_{n+m}]}_{\text{loop w/ homotopy class rel. }\partial} \overset{(*)}{=} [\omega_n]\cdot[\omega_m] \overset{(**)}{=} [\omega_n \cdot \omega_m]$

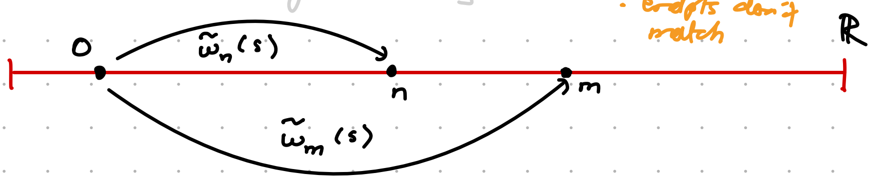
WTS same homotopy class $(*)$ & $(**)$.
↳ need to construct an explicit homotopy between these two paths...

Need to construct a homotopy $j_t(s)$ w/ $j_0(s) = \omega_{n+m}(s)$, $j_1(s) = \omega_n \cdot \omega_m(s)$

Idea: do this on the line $\mathbb{R}$, not the circle: it has advantages! want lifts of these paths ω_{n+m} and $\omega_n \cdot \omega_m$ to $\mathbb{R}$?

lifted our loops to paths.
But paths can't be concatenated ∵ endpts don't match

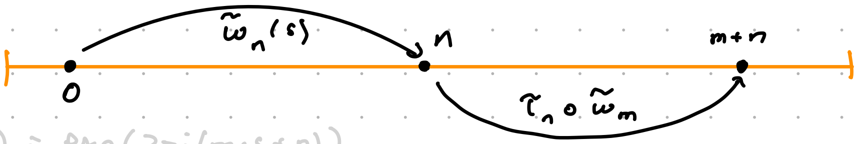
We have $\tilde{\omega}_{n+m}$ which lifts ω_{n+m}. We need a lift of $\omega_n \cdot \omega_m$.
What about $\tilde{\omega}_n \cdot \tilde{\omega}_m$ [concatenate the lifts of both]



Let $\tau_n : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\underbrace{\tau_n(r) = n + r}_{\text{just } \mathbb{Z} \text{ translations}}$$

claim: $\tau_n \circ \tilde{\omega}_m$ is a lift of ω_m.



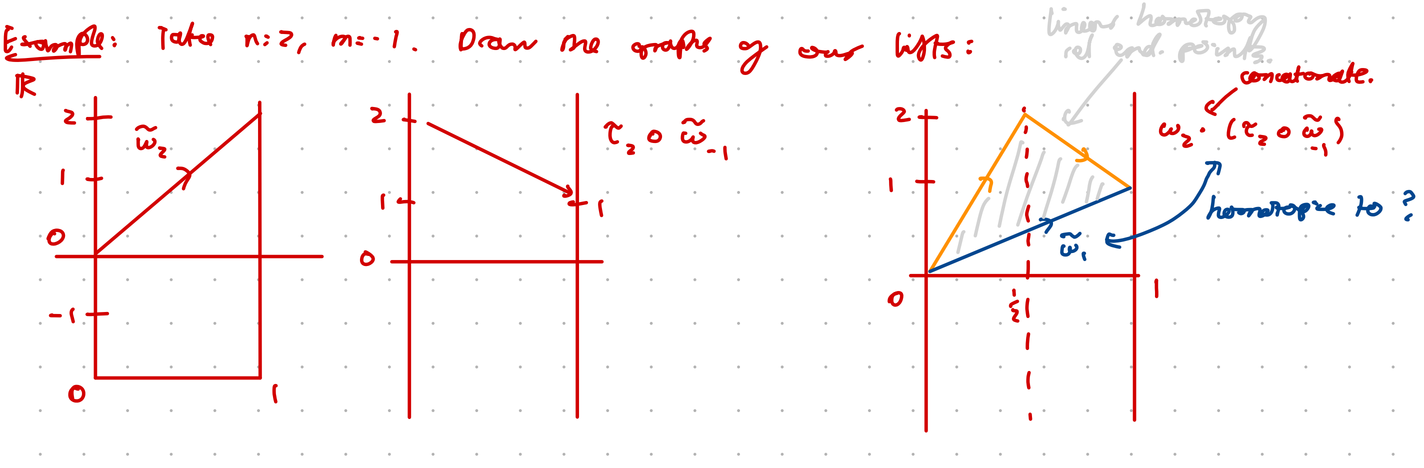
Proof: $p_\infty(\tau_n \circ \tilde{\omega}_m(s)) = p_\infty(m\cdot s + n) = \exp(2\pi i(m\cdot s + n))$

$= \exp(2\pi i m \cdot s)\underbrace{\exp(2\pi i n)}_{1}$ ← integer shift so no rotation

$= \exp(2\pi i m \cdot s) = \omega_m(s)$

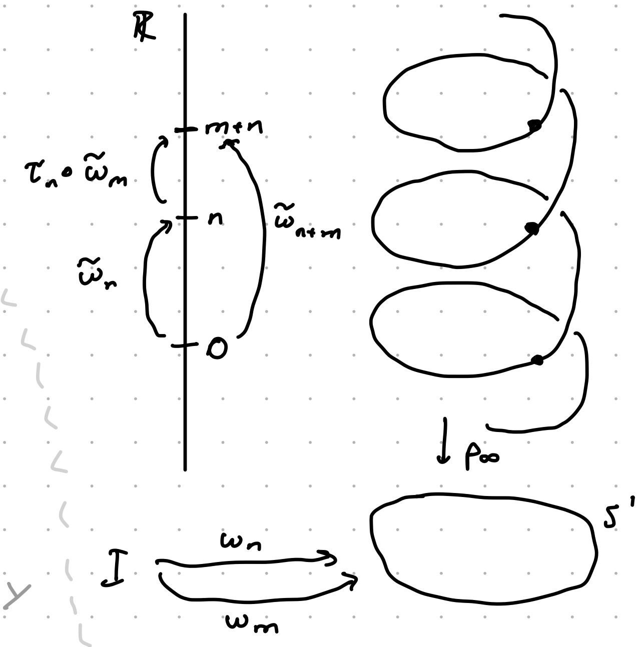
claim: $\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)$ is a lift of $\omega_n \cdot \omega_m$

Example: Take $n=2$, $m=-1$. Draw the graphs of our lifts:



$\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ $\Phi(n) = \omega_n$

showing Φ is a homomorphism,
specifically $\omega_n \cdot \omega_n \simeq \omega_{n+m}$ rel ∂



claim: we have a lift of $\omega_n \cdot \omega_m$ given by $\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)$

Pf:

① $\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)(0) = \tilde{\omega}_m(0) = 0$

Recall: $\tilde{\omega}_{m+n}(0) = 0$ $\tilde{\omega}_{m+n}(1) = m+n$

② **lemma:** say $g, h : I \to X$, $j : X \to Y$

Then $j \circ (g \cdot h) = (j \circ g) \cdot (j \circ h)$

Proof: say $0 \le s \le \frac{1}{2}$ ← LHS

Then $(j \circ g) \cdot (j \circ h)(s) = j \circ g(2s)$

say $\frac{1}{2} \le s \le 1$ ← RHS

$(j \circ g) \cdot (j \circ h)(s) = j \circ h(2s-1)$

This is the formula for $(j \circ g) \cdot (j \circ h)$

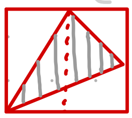
so done.

③ $\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)$ is a lift of $\omega_n \cdot \omega_m$

$p_\infty(\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)) = p_\infty(\tilde{\omega}_n) \cdot p_\infty(\tau_n \circ \tilde{\omega}_m) = \omega_n \cdot \omega_m$

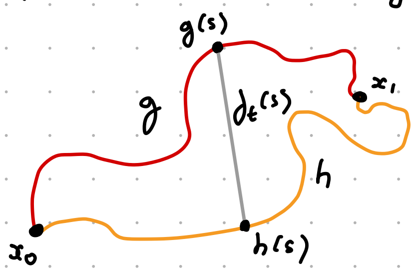
p_∞ does nothing to τ_n.

Linear Homotopies



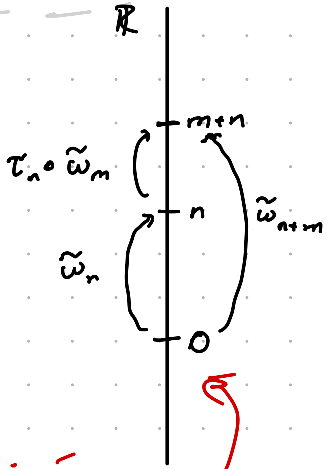
← These are the linear homotopies.

Picture in $\mathbb{R}^2$. Define $j_t(s) = (1-t)g(s) + t\,h(s)$



Note: $j_0(s) = g(s)$, $j_1(s) = j(s)$

$j_t(0) = x_0$, $j_t(1) = x_1$



Def: $\tilde{j}_t(s) = (1-t)\tilde{\omega}_{n+m} + t\,\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)$ Homotopy 'upstairs' fixing endpts.

Set $j_t(s) = p_\infty(\tilde{j}_t(s))$ ← Built a homotopy 'downstairs' just by pushing this down to S^1

What we wanted to show

① $j_0(s) = p_\infty \circ \tilde{j}_0(s) = p_\infty(\tilde{\omega}_{n+m}) = \omega_{n+m}$
② $j_1(s) = p_\infty \circ \tilde{j}_1(s) = p_\infty(\tilde{\omega}_n \cdot (\tau_n \circ \tilde{\omega}_m)) = \omega_n \cdot \omega_m$] homotopy

③ $j_t(0) = p_\infty \circ \tilde{j}_t(0) = p_\infty(0) = 1$
④ $j_t(1) = p_\infty \circ \tilde{j}_t(1) = p_\infty(m+n) = 1$] homotopy rel. endpoints

2nd proposition

Prop: $\Phi: \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ is surjective.

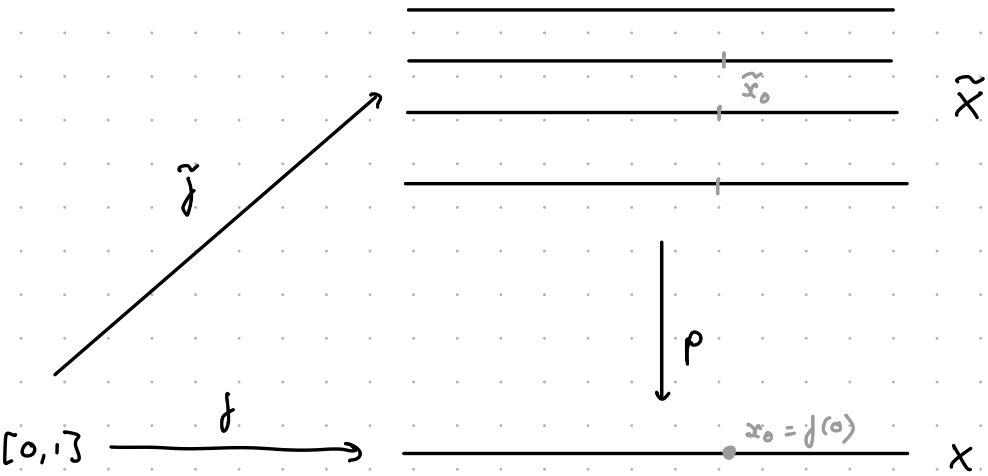
we've learnt: It's easier to reduce the problem to a vector space ($\mathbb{R}$)

Proof: Let $\gamma: [0,1] \longrightarrow S^1$, $\gamma(0) = \gamma(1) = 1$.

we just know this is cts. No obvious way to construct a lift $\Rightarrow$ Appeal to an existence result so we know a lift exists $\Rightarrow$ we need a lifting theorem for paths

Don't want γ want a lift of γ!

Lifting Theorem for Paths: Let $p: \tilde{X} \longrightarrow X$ be a covering map. Let $\gamma: [0,1] \longrightarrow X$ be a path starting at $x_0 \in X$. For each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{\gamma}: [0,1] \longrightarrow \tilde{X}$ starting at $\tilde{x}_0$



Proof: Later...

Let $\tilde{\gamma}: [0,1] \longrightarrow \mathbb{R}$ w/ $\tilde{\gamma}(0) = 0$ $\leftarrow$ get this by applying the lifting theorem

lift should measure total # of turns $\leftarrow$ intuition

$[\gamma$ is a loop$]$

Now, $p_\infty(\tilde{\gamma}(1)) = \gamma(1) = 1$

This is an integer!

If we have a loop downstairs, then total # of turns is an integer.

so $\tilde{\gamma}(1) \in p_\infty^{-1}(1) = \mathbb{Z}$

$\Rightarrow \tilde{\gamma}(1) = n$ for some $n \in \mathbb{Z}$

homotopic

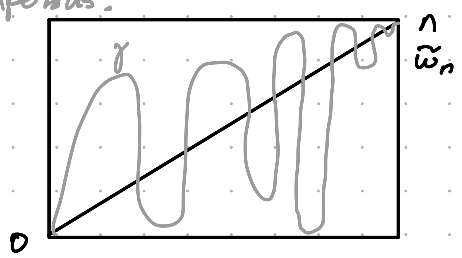
WTS $\tilde{\gamma} \simeq \omega_n$ rel. ∂

$\hookrightarrow$ motivation for lifting was to find an n.

$\leftarrow$ easier to build a homotopy upstairs.

Let $\tilde{\gamma}_t(s) = (1-t)\tilde{\omega}_n(s) + t\,\tilde{\gamma}(s)$

Define $\gamma_t(s) = p_\infty(\tilde{\gamma}_t(s))$, arguing as before that γ_t is a homotopy between ω_n and γ rel endpoints.



This crazy line is homotopic to a straight line path. Amazing!

Prop: $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$ is injective

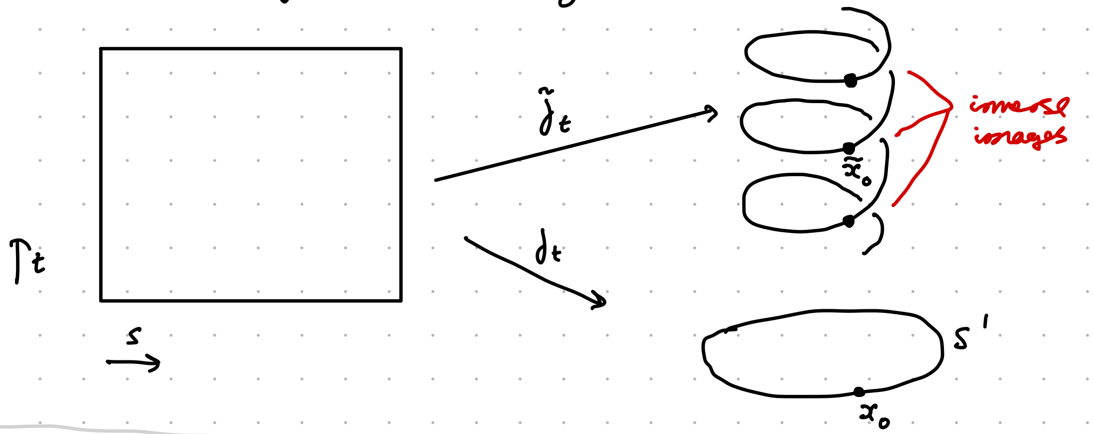
Take our problem on the circle & lift it up to $\mathbb{R}$

Proof: WTS if $w_n \simeq w_m$ rel ∂, then $m = n$

(we have a homotopy in the circle, don't have a lift.)

we're showing some fundamental group is not trivial: some homotopies can't exist. Given a path, there is no homotopy

Lifting theorem for homotopies: Let $j_t : I \longrightarrow X$ be a homotopy from j_0 to j_1, where $j_t(0) = x_0$. Let $\tilde{x}_0 \in p^{-1}(x_0)$. There is a unique lifted homotopy $\tilde{j}_t : I \longrightarrow \tilde{X}$, of paths starting at $\tilde{x}_0$ s.t. $p \circ \tilde{j}_t = j_t$



Proof: later...

As $w_n \simeq w_m$, say j_t is a homotopy rel ∂ between w_n and w_m

In part. $j_t(0) = j_t(1) = 1$

Now let $\tilde{j}_t : [0,1] \longrightarrow \mathbb{R}$ be a lift of j_t s.t. $\tilde{j}_t(0) = 0$

$\tilde{j}_0$ is a lift of w_n w/ $\tilde{j}_0(0) = 0$

$\tilde{w}_n$ is also a lift of w_n w/ $\tilde{w}_n(0) = 0$

lifts of same thing w/ same starting pt.

two paths both share same endpoints

By uniqueness of path lifting, $\tilde{j}_0(s) = \tilde{w}_n(s)$, equally, $\tilde{j}_1(s) = \tilde{w}_m(s)$

We consider $\tilde{j}_t(1)$. We have $\tilde{j}_0(1) = \tilde{w}_n(1) = 1$

Also, $\tilde{j}_1(1) = \tilde{w}_m(1) = 1$

Two loops homotopic $\Downarrow$ Same # of turns

But $\underbrace{p \circ (\tilde{j}_t(1))}_{\text{lift of a homotopy rel endpts}} = j_t(1) = 1 \Rightarrow \tilde{j}_t(1) \in p^{-1}(1) = \mathbb{Z}$

So, as a function of t, $\underbrace{\tilde{j}_t(s) \text{ is continuous and } \mathbb{Z} \text{ valued}}_{\text{MVT \& connectivity of } \mathbb{Z}} \Rightarrow \tilde{j}_t(1)$ constant.

In particular $\left.\begin{array}{l} \tilde{j}_0(1) = \tilde{w}_n(1) = n \\ \tilde{j}_1(1) = \tilde{w}_m(1) = m \end{array}\right] \Rightarrow m = n \Rightarrow \Phi$ is injective.

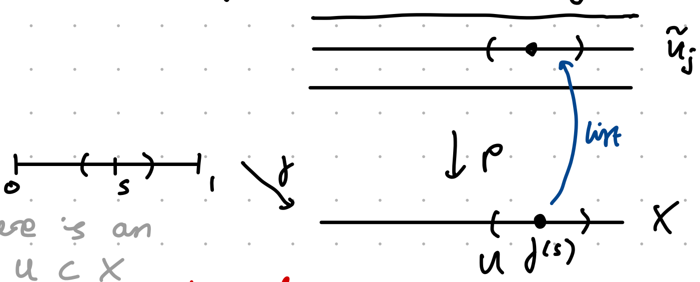
$\Rightarrow \pi_1(S^1)$ is a free abelian group generated by w_1

multiple of a single loop.

Lifting Theorem for paths:

Let $p: \tilde{X} \longrightarrow X$ be a covering map. Let $\gamma: [0,1] \longrightarrow X$ be a path starting at $x_0 \in X$. Let $\tilde{x}_0 \in p^{-1}(x_0)$. Then $\exists$ unique lift $\tilde{\gamma}: I \longrightarrow \tilde{X}$ starting at $\tilde{x}_0$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$



$p|U_j$ is a homeomorphism so $\exists$ an inverse to $p|\tilde{U}_j$

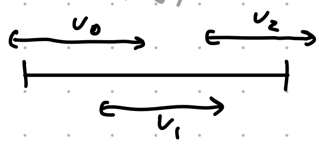
no problems w/ local inverses...

Proof: for each $s \in [0,1]$, there is an evenly covered open set $U \subset X$ containing $\gamma(s)$

he made notation mistakes here... careful...

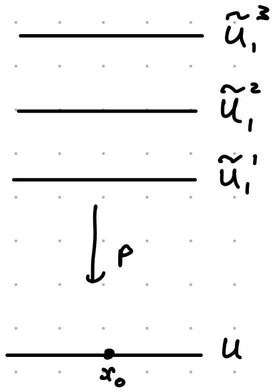
By compactness of $[0,1]$, $\exists$ finite collection $V_0, \dots, V_n$ which cover $[0,1]$, so we have

$$0 = t_0 < t_1 < \dots < t_n < 1 \quad \text{so that} \quad [t_k, t_{k+1}] \subset V_k$$



And $\gamma([t_k, t_{k+1}]) \subset U_k$

Construct $\tilde{\gamma}$ on $[0, t_1]$. $\gamma(0) = x_0$, $p[\tilde{x}_0] = x_0$



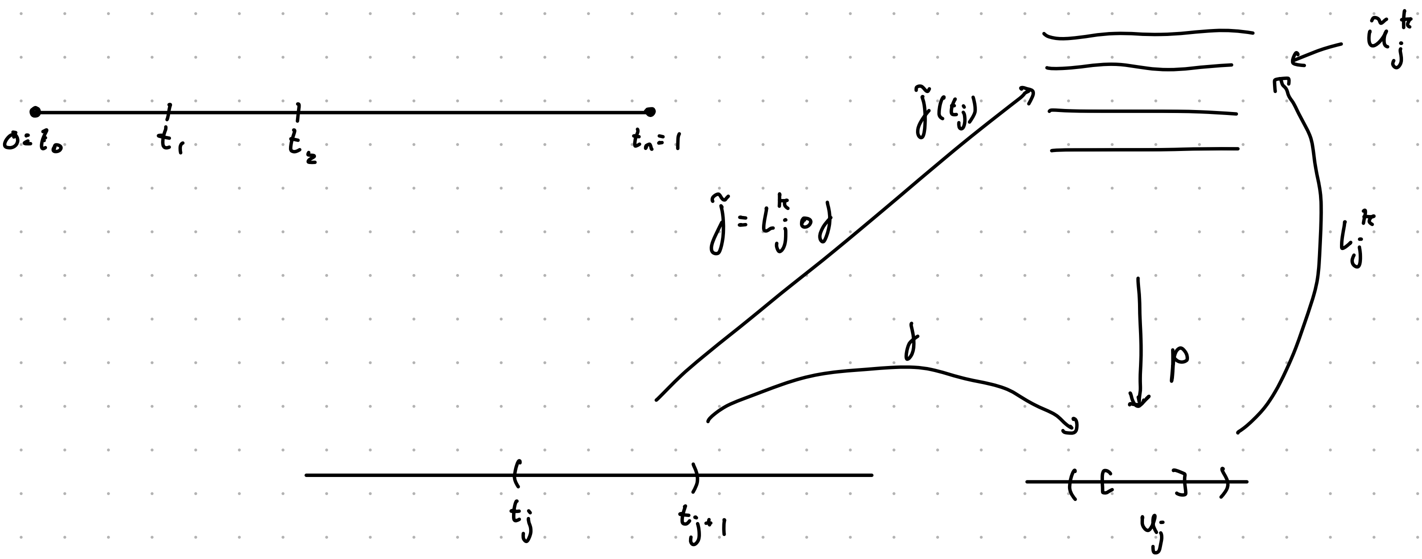
Let $L^j: U_1 \longrightarrow \tilde{U}_1^j$ be the inverse to $p|\tilde{U}_1^j$, define $\tilde{\gamma}$ on $[0, t_1]$ by $\tilde{\gamma} = L^j \circ \gamma$

Continue inductively

<u>Lecture ?</u> 23/10/23

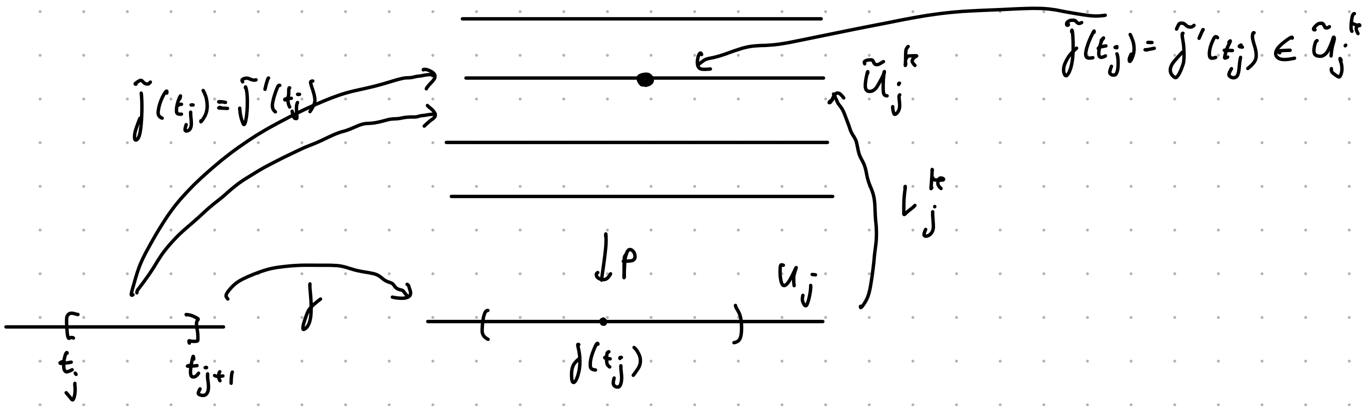
Proof of uniqueness

$$\gamma([t_j, t_{j+1}]) \subset U_j \subset X$$



Assume we have two lifts $\tilde{\gamma}, \tilde{\gamma}'$ with $\tilde{\gamma}(0) = \tilde{\gamma}'(0) = x_0$ — assuming agree at left endpoint & disagree elsewhere

Say $\tilde{\gamma} = \tilde{\gamma}'$ for $s \in [0, t_j]$ for $t_j < 1$ and $\tilde{\gamma} \neq \tilde{\gamma}'$ for $[t_j, t_{j+1}]$



- $[t_j, t_{j+1}]$ is connected

  $\Rightarrow$ $\gamma([t_j, t_{j+1}])$ is connected as γ cts

- $\tilde{\gamma}'([t_j, t_{j+1}])$, $\tilde{\gamma}([t_j, t_{j+1}])$ both connected

  $\Rightarrow$ Conclude, both sets lie in the same inverse image of U_j, $\tilde{U}_j^k$.

  γ & γ' agree at $\gamma(t_j)$ so must both lie in same set

Claim: $\tilde{\gamma}$ and $\tilde{\gamma}'$ restricted to $[t_j, t_{j+1}]$ are both given by $\underbrace{L_j^k}_{\text{local lift}} \circ \underbrace{\gamma}_{\text{function}}$.

eye-oh-ta

Formally, recall that L_j^k is a homeomorphic inverse to p so

restricted

$$L_j^k \circ p \,|\, \tilde{U}_i^j = Id_{\tilde{U}_i^j}, \qquad p \circ L_j^k \,|\, U_j = Id_{U_j}$$

tells us that $L_j^k \circ \gamma$ is a lift: $p \circ L_j^k \circ \gamma = \gamma$

these images contained in this set

Since, $\tilde{\gamma}([t_j, t_{j+1}]) \subset \tilde{U}_j^k$

expand identity — so just γ

$$\underbrace{\tilde{\gamma}\,|\,[t_j, t_{j+1}]}_{} = Id_{\tilde{U}_j^k} \circ \tilde{\gamma}\,|\,[t_j, t_{j+1}] = L_j^k \circ \overbrace{p \circ \tilde{\gamma}}^{\text{lift}} = \underbrace{L_j^k \circ \gamma}_{\text{no } \tilde{\gamma} \text{ dependence?}}$$

our lift $\tilde{\gamma}$ is given by

$\tilde{\gamma}'$ " $\tilde{\gamma}'$ " $\tilde{\gamma}' = L_j^k \circ \gamma$

these are equal

$\Rightarrow$ have uniqueness! ☒

General Lifting Theorem

parametrised version of the lifting thm for paths → Y is a general parameter space

Let $p: \tilde{X} \to X$ be a covering map.

Given $F: \underbrace{[0,1]}_{s} \times \underbrace{Y}_{y} \to X$ and a map $\tilde{F}_0 : Y \to \tilde{X}$ lifting $F|\{0\}\times Y$ $(p \circ \tilde{F}_0(Y) = F(Y,0))$, then there is a unique map $\tilde{F}: [0,1]\times Y \to \tilde{X}$ lifting F $(p\circ \tilde{F} = F)$ which is given by $\tilde{F}_0(Y)$ on $\{0\}\times Y$

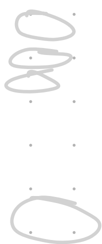
path parameter → s

y ← homotopy parameter

was t before but a more general space now.

$y = t \in [0,1]$

$$\begin{tikzcd}
\{0\}\times Y \arrow[r, "\tilde{F}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\
I \times Y \arrow[r, "F"] \arrow[ur, "\tilde{F}"] & X
\end{tikzcd}$$



proving in a more general context

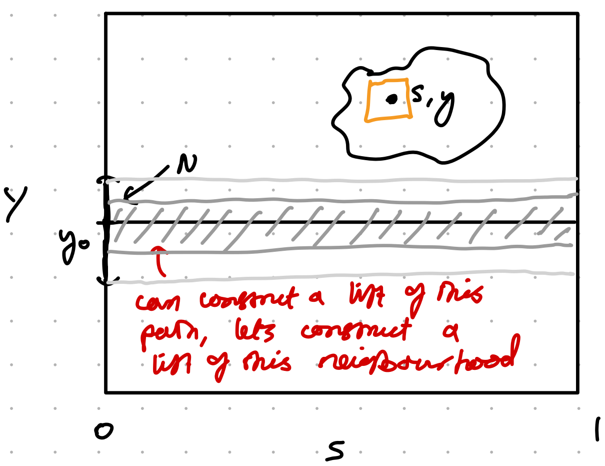
Proof: Using our path lifting theorem, we can lift each path $F: [0,1]\times Y \to X$ to a path $\tilde{F}: [0,1]\times Y \to \tilde{X}$ w/ $\tilde{F}((0,y)) = \tilde{F}_0(y)$

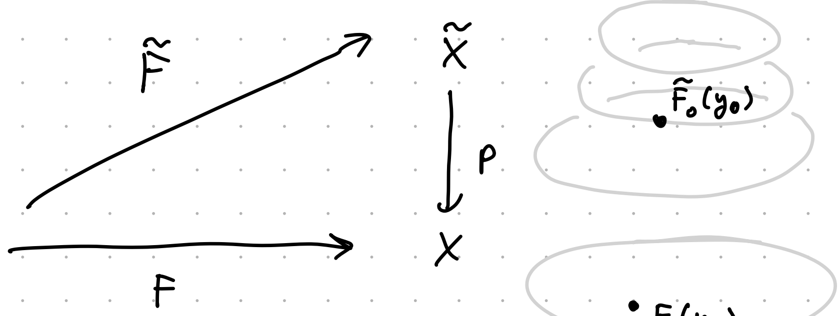
Thus, there is a function $\tilde{F}$ which satisfies the thm. Doesn't tell us that this function is continuous. (Unique path lifting)

$$\begin{tikzcd}
& \tilde{X} \arrow[d, "p"] \\
I \times \{y\} \arrow[r] \arrow[ur, "\tilde{F}"] & X
\end{tikzcd}$$

WTS function cts (we have uniqueness on every path so showing cty ⇒ uniqueness too)

Pick a $y_0 \in Y$. We will construct a cts lift of a family of paths $N \times [0,1]$ with $N \subset Y$





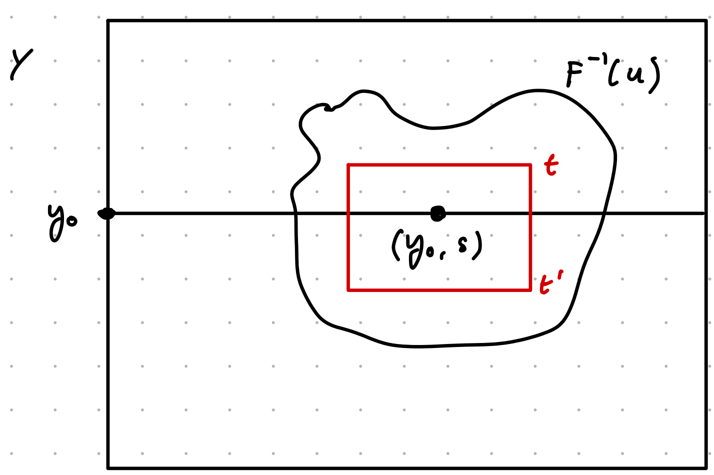
Given $(s,y) \in [0,1]\times Y$

$F((s,t))$ is contained in any evenly covered set $U \subset X$

$\Rightarrow \exists\, (t,t') \times N$ mapping to U,

Lecture 11 (?) 24/10/23

We have a function $\tilde{F}$ that lifts F and satisfies the initial condition y_0. WTS this function is continuous. Need to find a lift on some neighbourhood of y_0 to prove cty.



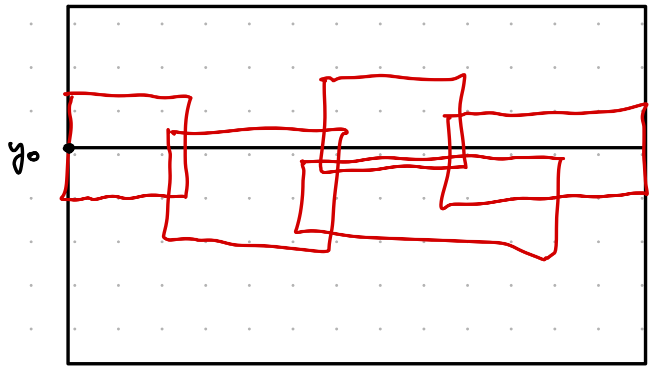
Pick $y_0 \in Y$. Want to find some neighbourhood N of y_0 and a lift of $F|[0,1]\times N$

For any (y_0, s), $s \in [0,1]$,

$$F((y_0, s)) \subset U,$$

U evenly covered and open.

We can find a neighbourhood $(t, t') \times N$ containing (y_0, s)

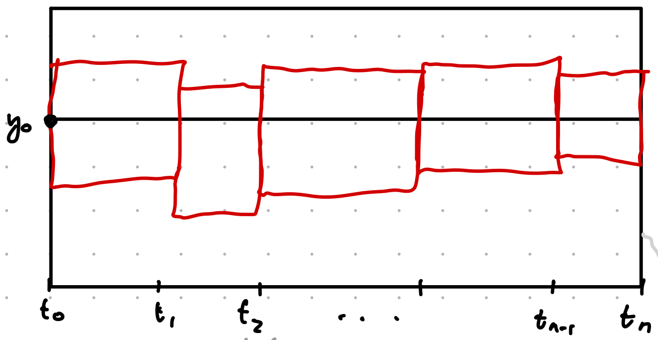


Using compactness of $I = [0,1]$, there is a finite set of product neighbourhoods covering $[0,1] \times \{y_0\}$

We can find a $0 = t_0 < t_1 < \dots < t_n = 1$ so that $[t_j, t_{j+1}] \times N_j$ cover the interval and

$$F([t_j, t_{j+1}] \times N_j) \subset U_j$$

w/ U_j evenly covered.



Define a lift by setting

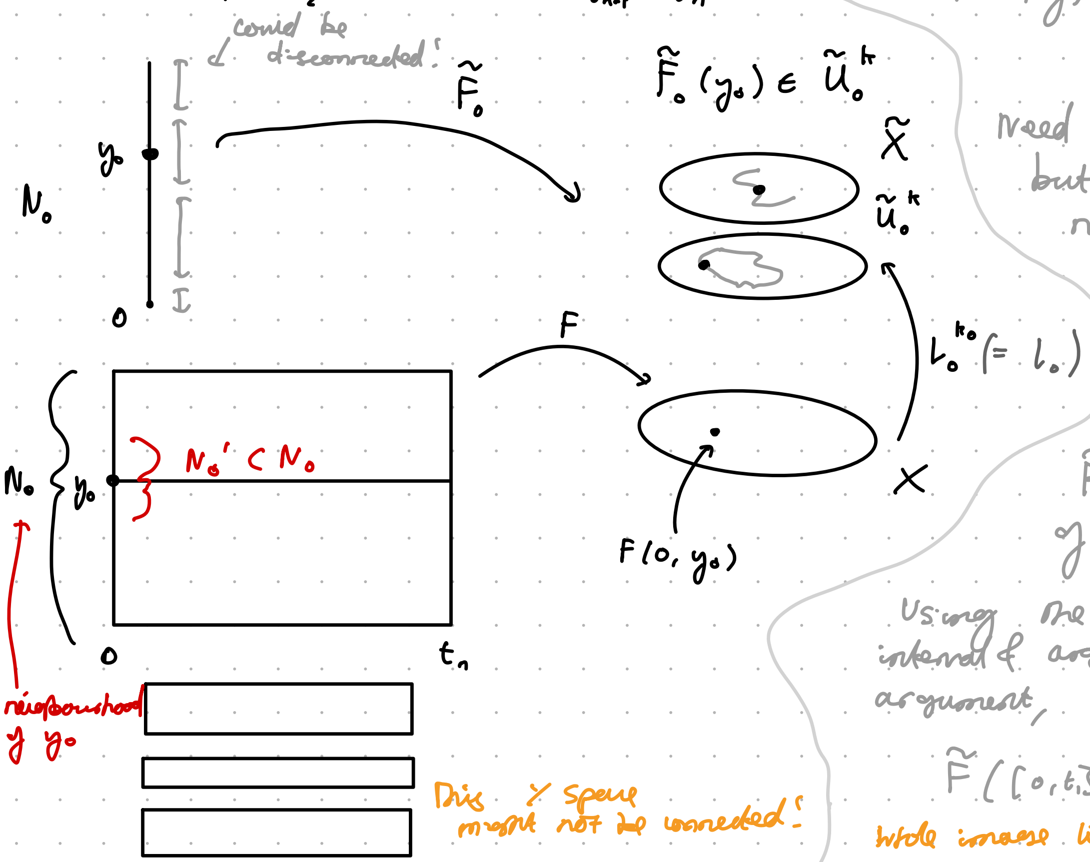
$$\tilde{F}(s, y) = L_0 \circ F(s, y)$$

lots of different lifts so does $\tilde{F}$ satisfy initial conditions?

ie, does

$$\tilde{F}(0, y) = \tilde{F}_0(y) \quad ???$$

(ok for $y = y_0$)



Need not be true in general, but there is some smaller neighbourhood $N_0' \subset N_0$ so that

$$\tilde{F}_0(N_0') \subset \tilde{U}_0^{k_0}$$

using cty of $\tilde{F}_0$ and openness of $\tilde{U}_0^{k_0}$

Using the connectivity of the interval & arguing as in path lifting argument, we get

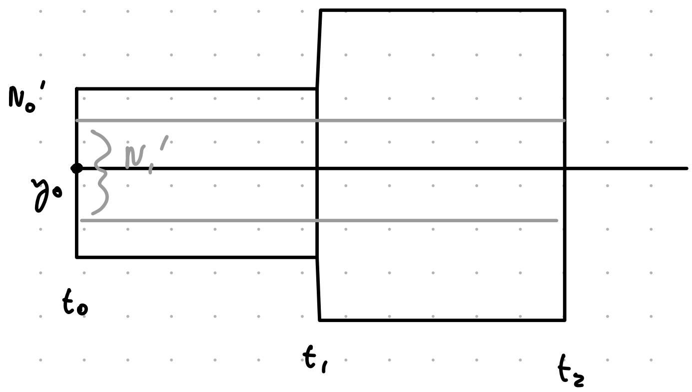
$$\tilde{F}([0, t_1] \times N_0') \subset \tilde{U}_0^{k_0}$$

Whole image lies in our set.

Thus $\tilde{F}([0,t_1]\times N_0')$ is the lift that satisfies the initial conditions. It is continuous since it is equal to $L_0 \circ F$.

Now consider $[t_1, t_2] \times (N_0' \cap N_1)$. we want a lift $\tilde{F}$ defined on $[t_1,t_2]\times(N_0'\cap N_1)$ so that it agrees w/ the previous lift on

$$\{t_1\} \times (N_0' \cap N_1)$$



As before, there's some neighbourhood

$N_1' \subset N_0' \subset N_0$ satisfying

$\tilde{F}(t,y)$ lifts to $\tilde{U}_1^{k_1}$

for $y \in N_1'$

Define $\tilde{F}$ to be $L_1 \circ F$ on this set.

$\Rightarrow$ we have $\tilde{F}$ defined on $[0,t_1]\times N_1'$ and $[t_1,t_2]\times N_1'$

They agree on the overlap. By the pasting lemma, we have a [cont. ?] function on $[0,t_2]\times N_1'$.

Repeat this for the remaining intervals gives a cts lift on $[0,1]\times N_n'$.
[only doing this a finite # times, so all allowed.] ∎

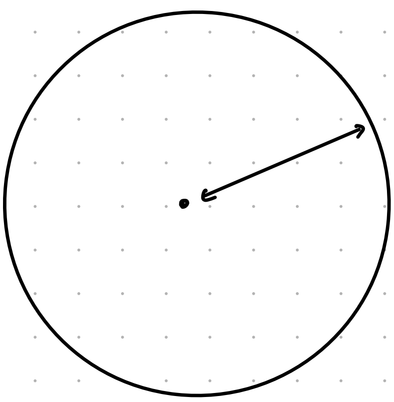
Lecture 12 24/10/23

Applications

$\boxed{\text{The fundamental thm of algebra: Every non-constant polynomial w/ coefficients in } \mathbb{C} \text{ has a root in } \mathbb{C}.}$

Proof: Say $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ is a polynomial of degree $n \geq 0$.

Suppose p has no roots. WT consider values of p on circle of radius r.
write $s \longmapsto re^{2\pi i s}$ to parametrise the circle.



We get a map to the circle by looking at $\frac{p(z)}{|p(z)|}$ [well defined & cts $\because$ assume no roots]

We write $f_r(s) = \frac{p(re^{2\pi i s})}{|p(re^{2\pi i s})|}$

For each r, we have a map to S.

$f_r(0) = f_r(1)$ so a loop!

We can further normalise: let $\bar{f}_r(s) = \dfrac{p(re^{2\pi i s})}{|p(re^{2\pi i s})|} \cdot \Big(\underbrace{\dfrac{p(r)}{|p(r)|}}_{\text{value at zero to normalise}}\Big)^{-1}$

With this, we have $\bar{f}_r(0) = \bar{f}_r(1) = 1$

If $r = 0$, then $\bar{f}_r(s) = 1 = \omega_0(s)$

What happens when $r \longrightarrow \infty$?   behaviour of z^n dominates... WTS $r \to \infty \Rightarrow$ map around circle n times.

Fix a value of r_0 sufficiently large

$$r_0 > \{|a_1| + |a_2| + \dots + |a_n| \text{ and } 1\}$$

Intuition: path goes around circle n times $\Rightarrow$ find a homotopy!

Consider $p_t(s) = \underbrace{z^n + t(a_1 z^{n-1} + \dots + a_n)}_{\text{linear homotopy!}}$ $[\,p_0(s) = z^n = \underbrace{e^{2\pi i n s}}_{\omega_n(s)}$ circles! $,\ p_1(s) = p(z)\,]$

For fixed z, linear map

Polynomials $\longrightarrow$ paths now

Consider $g_t(s) = p_t(r_0 e^{2\pi i s})$ & normalise to unit circle

$$\bar{g}_t(s) = \frac{p_t(r_0 e^{2\pi i s})}{|p_t(r_0 e^{2\pi i s})|} \underbrace{\frac{|p_t(r_0)|}{p_t(r_0)}}_{\text{start at } 1}$$

$\bar{g}_t(s): [0,1] \longrightarrow S^1 \subset \mathbb{C}$

$\bar{g}_t(0) = \bar{g}_t(1) = 1$

Claim $\bar{g}_t(s)$ is a cts function for $0 < t < 1$. For $|z| = r_0$

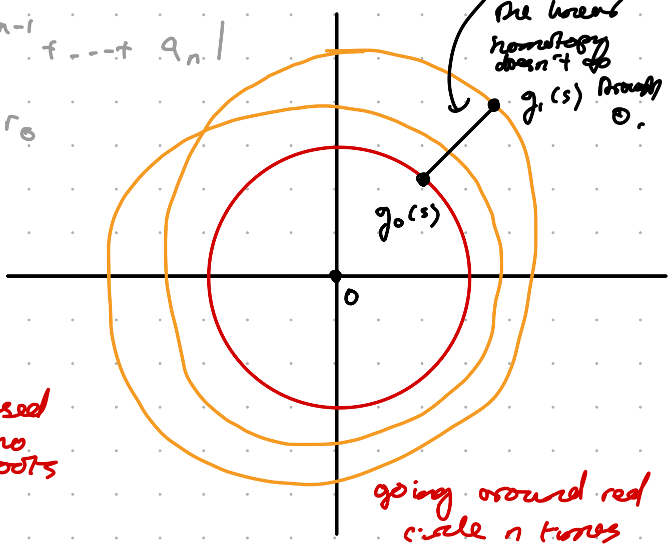
(Need to know that $p_t(z)$ has no zeros for $|z| = r_0$)

$$|z^n| > |z^{n-1}| \underbrace{(|a_1| + \dots + |a_n|)}_{|z| = r_0} \geq |a_1 z^{n-1}| + |a_2 z^{n-2}| + \dots + |a_n|$$

$$\geq |a_1 z^{n-1} + \dots + a_n|$$

This calc shows the linear homotopy doesn't go through 0.

Thus $p_t(z)$ has no roots for $|z| = r_0$



Now: Two homotopies, $\bar{f}$ and $\bar{g}$.

As t goes from 0 to r_0

① $\bar{f}_t$ gives a homotopy between the constant path $\bar{f}_0(s) = 1 = \omega_0(s)$ and $\bar{f}_{r_0}$ } used no roots

② As t goes from 1 to 0, $\bar{g}(s)$ gives a homotopy between

$\bar{f}_{r_0}(s)$ and $\bar{g}_1(s) = e^{2\pi i n s} = \omega_n(s)$ } used r_0 large enough

Putting these two homotopies together, we get a homotopy rel endpoints from ω_0 to ω_n.

conclude $n=0$ as $\pi_1(S^1, 1) = \mathbb{Z}$

$\Rightarrow$ $p(z)$ is a constant polynomial. ■

<u>Monday 30th October 2023</u>

Write (X, x_0), X top. space. $x_0 \in X$

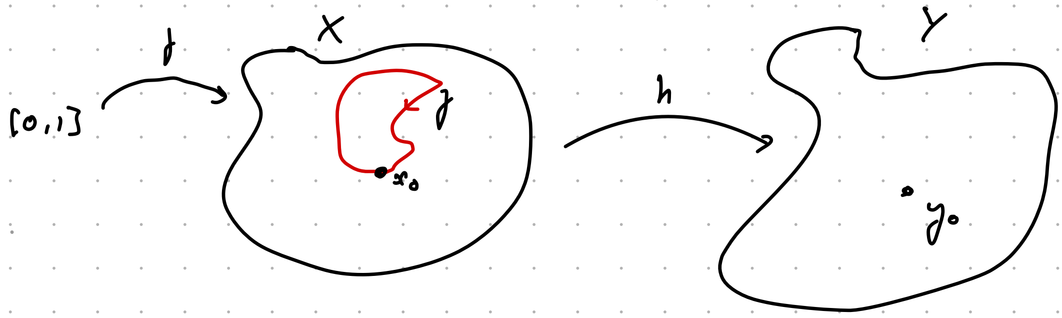
If $h: X \to Y$ w/ $h(x_0) = y_0$, write $h: (X, x_0) \to (Y, y_0)$

If $h: (X, x_0) \to (Y, y_0)$, then there is an induced function

$$h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

<u>Proof:</u>

missed this; See LN...



<u>Example</u> let $p_2: \underset{\mathbb{C}}{\overset{\frown}{S^1}} \to S^1_{\subset\mathbb{C}}$ be given by $p_2(z) = z^2$

claim that $(p_2)_*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is given by $\omega_n \mapsto \omega_{2n}$

$$\begin{array}{ccc} \pi_1(S^1,1) & \longrightarrow & \pi_1(S^1,1) \\ \wr\wr & & \wr\wr \\ \mathbb{Z} & & \mathbb{Z} \\ n & \longmapsto & 2n \end{array}$$

check: $(p_2)_*([\omega_n]) = [p_2 \circ \omega_n(s)] = [\exp$?

<u>Lemma:</u> The induced homomorphism satisfies

(1) $(Id_{(X,x_0)})_* = Id_{\pi_1(X,x_0)}$

(2) If $f: (X, x_0) \to (Y, y_0)$ and $g: (Y, y_0) \to (Z, z_0)$, then $(g \circ f)_* = g_* \circ f_*$

<u>Proof:</u> 1 is clear.

(2) $(g \circ f)_*(\underbrace{[\gamma]}_{\text{homotopy class of some loop gamma}}) = [g \circ f \circ \gamma] = g_*([f \circ \gamma]) = (g_* \circ f_*)([\gamma])$

Turns into a Q about groups of homomorphisms.

Thm: If $f:(X,x_0)\longrightarrow(Y,y_0)$ is a homeomorphism, then the induced map
$f_*:\pi_1(X,x_0)\longrightarrow\pi_1(Y,y_0)$ is a group isomorphism.

spaces
$\Downarrow$
groups!

Proof: $\mathrm{Id}_{(X,x_0)} \circlearrowright (X,x_0) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} (Y,y_0) \circlearrowleft \mathrm{Id}_{(Y,y_0)}$

by lemma

$$\mathrm{Id}_{\pi_1(X,x_0)} = (\mathrm{Id}_{(X,x_0)})_* = (f^{-1}\circ f)_* = (f^{-1})_*\circ(f)_*$$

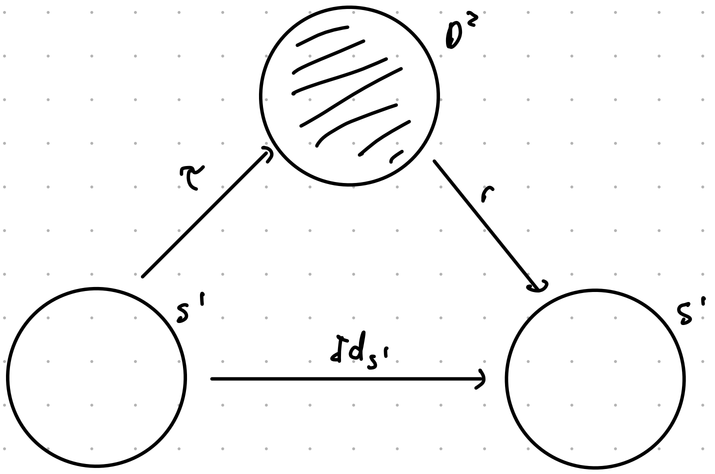
$$\mathrm{Id}_{\pi_1(Y,y_0)} = (f_*)\circ(f^{-1})_* \qquad \{\text{same logic}\}$$

$\Rightarrow$ f_* is an isomorphism (left & right inverse) ■

Example

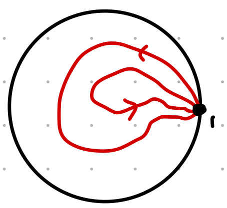
Let $\iota: S^1\longrightarrow D^2$ be the inclusion of the boundary

Let $r: D^2\longrightarrow S^1$ be a retraction, that is $r\circ\iota=\mathrm{Id}_{S^1}$



$$\begin{tikzcd}
& \pi_1(D^2,1) = \{0\} \arrow[dr, "r_*"] & \\
\underset{\mathbb{Z}}{\pi_1(S^1,1)} \arrow[ur, "\iota_*"] \arrow[rr, "\mathrm{Id}_{\pi_1(S^1,1)}"'] & & \underset{\mathbb{Z}}{\pi_1(S^1,1)}
\end{tikzcd}$$

Note: $\pi_1(D^2,1)$ is trivial since linear homotopies show every loop is homotopic to the constant loop at 1.



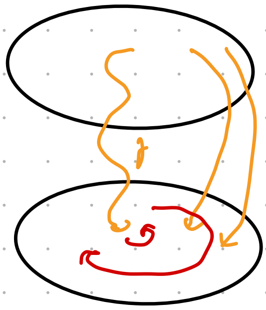
There is no such commutative diagram for groups & homomorphisms.

There is no retract from $D^2\longrightarrow S^1$.

maps, hard to know when they exist $\longrightarrow$ groups & homomorphisms much easier to say when they exist

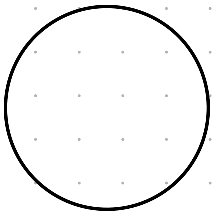
How topology works...

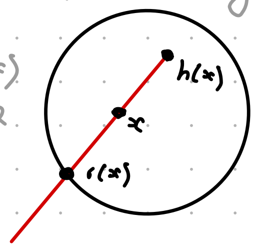
Thm: Every cts map $h: D^2 \longrightarrow D^2$ has a fixed point $x \in D^2$. $(h(x) = x.)$



However your map from upstairs to downstairs, there must be a fixed point.

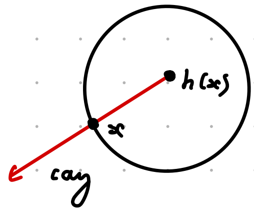
Proof: Assume h has no fixed point. Since $x \neq h(x)$, they determine a ray



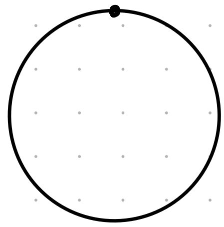


let $r(x)$ be the point where the ray intersects S^1 = boundary of D^1.

If $x \in S^1$, then $r(x) = x$.



Thus, $r: D^2 \longrightarrow S^1$ w/ $r \circ \underbrace{i}_{\text{inclusion map.}} = Id_{S^1}$



But no such retraction exists
$\Rightarrow$ have created an impossible object.

Borsuk Ulam theorem

For any cts map $f: S^2 \longrightarrow \mathbb{R}^2$, there exists a pair of antipodal points x and $-x$ in S^2 w/ $f(x) = f(-x)$

① $\Rightarrow$ There is no embedding of S^2 in $\mathbb{R}^2$ $\Leftarrow$ (won't be injective)

② $\Rightarrow$ At any given time, there are two antipodal points on the earth at which the $\underbrace{\text{temperature \& humidity}}_{\text{cts on } \mathbb{R}^2}$ are the same.

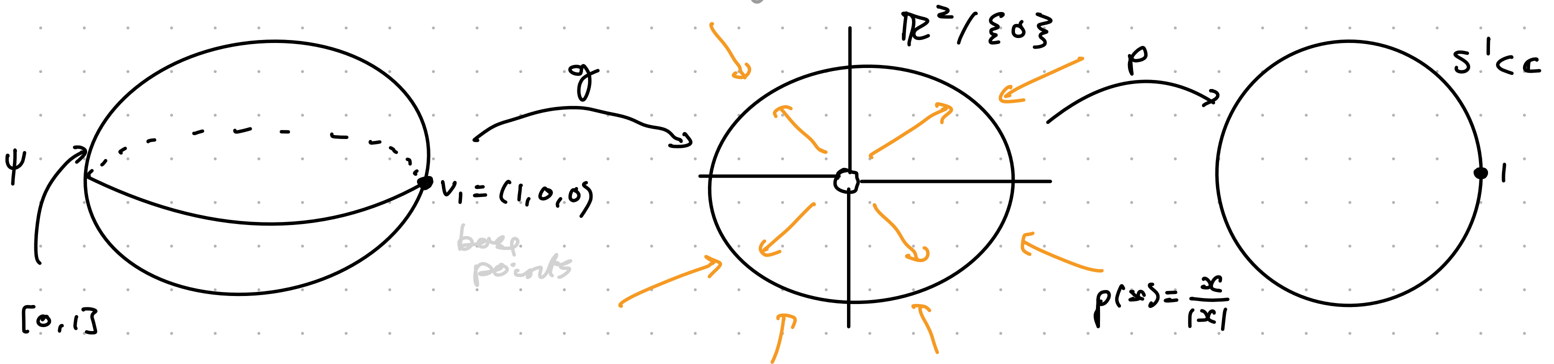
Proof: Assume there is an f w/ $f(x) = f(-x)$ for any $x \in S^2$

Define a function $g: S^2 \longrightarrow \mathbb{R}^2$ by $g(x) = f(x) - f(-x)$

By assumption, $g(x) \neq 0$ at $x \in S^2$.

Note that $g(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x)$, call such a function odd. [like trig]

Need to translate to topological language... How do we do this?



p is a deformation retract to S^1 [$\because \mathbb{R}^2/\{0\}$ so fine]
p is also an odd function

Hence, $p \circ g$ is also an odd function.

Parametrise the equator:

$$\psi(s) = (\cos(2\pi s), \sin(2\pi s), 0)$$

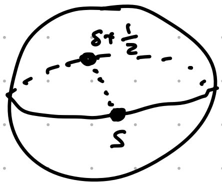
Need to send basepoints to each other so normalise,

Let $p_1(x) = \dfrac{p(x)}{p(g(v_1))}$ then $p_1(g(v_1)) = 1$ [rotation]

<u>Note:</u> $p_1 \circ g : (S^2, v_1) \longrightarrow (S^1, 1)$ [homeomorphism]

$$\underbrace{(p_1 \circ g)_*}_{\text{what does the induced map do to the equator?}}[\psi] \in \pi_1(S^1, 1)$$ [induced map]

What about oddness? Antipodal point on the equator?



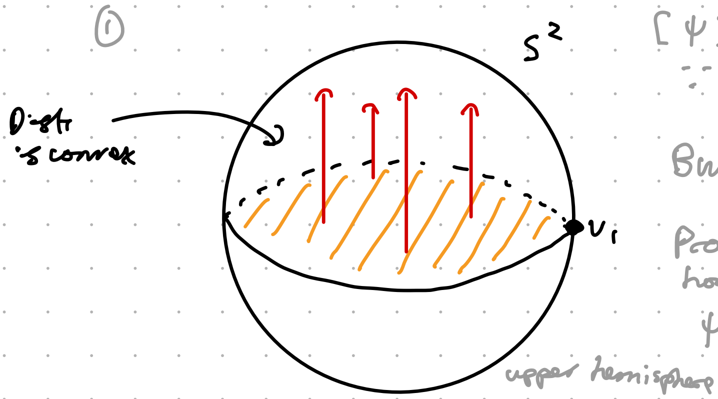
$$\underbrace{\psi(s + \tfrac{1}{2}) = -\psi(s)}_{\text{antipodal point on equator...}}$$

what oddness translates too

$$p_1 \circ g \circ \psi(s + \tfrac{1}{2}) = -p_1 \circ g \circ \psi(s)$$

<u>First Calculation of $(p_1 \circ g)_*[\psi]$:</u> (WTS loop is trivial)

①



$[\psi]$ is trivial in $\pi_1(D^2, v_1)$
$\because$ The disk is convex.

But D^2 is <u>not</u> on S^2. How get there?

Project! Define $\phi : D^2 \longrightarrow S^2$ by
homeomorphically...

$$\psi(x, y) = (x, y, \underbrace{\sqrt{1 - x^2 - y^2}}_{z \text{ value to put on the sphere!}})$$

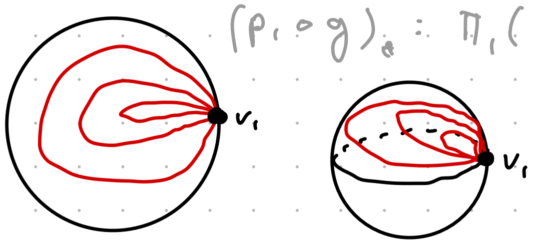
$\psi : (D^2, v_1) \longrightarrow (UH, v_1)$ [homeomorphism, basepoint preserving]

$\psi_* : \pi_1(D^2, v_1) \longrightarrow \pi_1(UH, v_1)$ is an isomorphism

So $[\psi]$ is trivial in $\pi_1(D^2, v_1) \implies [\psi]$ trivial in $\pi_1(UH, v_1)$

$(p_1 \circ g)_* : \pi_1(UH, v_1) \longrightarrow \pi_1(S^1, 1)$, takes $\overset{\text{class}}{[\psi]}$ to $\overset{\text{class}}{[0]}$

trivial element in $\pi_1(S^1, 1)$



So

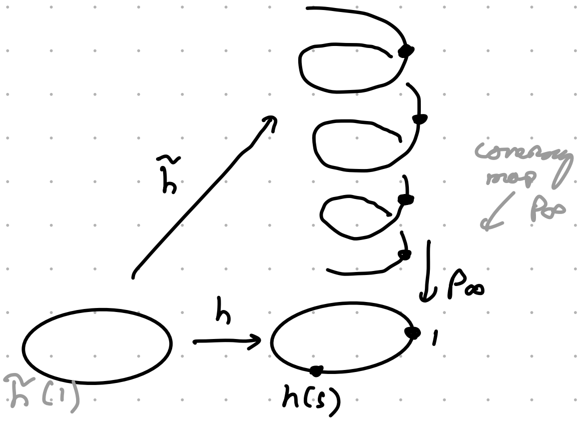
the constant path

$$(p_1 \circ g)_*[\psi] \sim [\omega_0] \in \pi_1(S^1, v_1)$$

Second calculation of $(p_1 \circ g)_* [\psi]$:

Will use the oddness to show that $p_1 \circ g \circ \psi(s)$ is non-trivial.

Write $h(s) = p_1 \circ g \circ \psi(s)$



oddness $\Rightarrow$ $h(s+\frac{1}{2}) = -h(s)$

By lifting thm, $\exists\, \tilde{h} : [0,1] \longrightarrow \mathbb{R}$ s.t.

$p_\infty \circ \tilde{h} = h$, $\quad \tilde{h}(0) = 0$

$\tilde{h}(1) \in \mathbb{Z}$ and $h \simeq \omega_m$ where $m = \tilde{h}(1)$

$\tilde{h}$ is mysterious, but we know

$$p_\infty \circ \tilde{h}(s+\tfrac{1}{2}) = -\, p_\infty \circ \tilde{h}(s)$$

We do know some other real number mapping to $-\,p_\infty(\tilde{h}(s))$

$$\begin{aligned} p_\infty(\tilde{h}(s) + \tfrac{1}{2}) &= \exp(2\pi i(\tilde{h}(s) + \tfrac{1}{2})) \\ &= \exp(2\pi i\, \tilde{h}(s) + \pi i) \\ &= \exp(2\pi i\, \tilde{h}(s)) \exp(\pi i) \\ &= -\, p_\infty(\tilde{h}(s)) \end{aligned}$$

$\Rightarrow$ Two different points mapping to same pt in circle.

Conclude: Since any r, r' w/ the same image under p_∞ differ by an integer (going up the helix in steps), we have

$$\tilde{h}(s+\tfrac{1}{2}) - (\tilde{h}(s) - \tfrac{1}{2}) = n_s \in \mathbb{Z}$$

Apriori, n_s depends on s, but LHS cts function of s so $n = n_s$ is constant.

$$\tilde{h}(s+\tfrac{1}{2}) = \tilde{h}(s) + \tfrac{1}{2} + n$$

Take $s = 0$, $\Rightarrow$ $\tilde{h}(\frac{1}{2}) = \underbrace{\tilde{h}(0)}_{=0} + \frac{1}{2} + n = n + \frac{1}{2}$

Take $s = \frac{1}{2}$ $\Rightarrow$ $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + n + \frac{1}{2}$

Substitute: $\tilde{h}(1) = n + \frac{1}{2} + n + \frac{1}{2} = 2n+1$

$\Rightarrow$ $h \simeq \omega_{2n+1}$

But 1st calc was $h \simeq \omega_0$

Thus $2n+1 = 0$, 0 even, $2n+1$ odd $\times$

{map w/out ~~the~~ this property & maths breaks}

<u>Proposition:</u> $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$

<u>Proof:</u> RHS is product of groups. A basic property of groups is that a function f is determined & determines a pair of functions g, h s.t. $f = (g, h)$

$$\begin{tikzcd}
 & G \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[d, dashed, "f"] & \\
\pi_1(X, x_0) & \pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[l] \arrow[r] & \pi_1(Y, y_0)
\end{tikzcd}$$

Basepoints still there, just removed for ease...

using the induced map

$$\begin{tikzcd}
 & \pi_1(X \times Y) \arrow[dl, "(\pi_x)_*"'] \arrow[dr, "(\pi_y)_*"] \arrow[d, "F"] & \\
\pi_1(X) & \pi_1(X) \times \pi_1(Y) \arrow[l] \arrow[r] & \pi_1(Y)
\end{tikzcd}$$

Note that $X \times Y \longrightarrow X$ gives an induced map from $\pi_1(X \times Y)$ to $\pi_1(X)$.

This is a homomorphism $F = ((\pi_x)_*, (\pi_y)_*)$

A basic property of the product topology is that a function $f: Z \longrightarrow X \times Y$ is cts iff the maps $g: Z \to X$, $h: Z \to Y$ defined by $f(z) = (g(z), h(z))$ are both cts

$$\begin{tikzcd}
 & Z \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[d, "f"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Take $Z = I$ & we get a way to construct loops.

<u>Claim:</u> F is surjective

<u>Pf:</u> Pick

construct paths

$([\gamma_x], [\gamma_y]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Then, (γ_x, γ_y) is a loop in $X \times Y$ which maps to γ_x under π_X and γ_y under π_Y.

This shows surjectivity.

$$\begin{tikzcd}
 & I \arrow[dl, "\gamma_x"'] \arrow[dr, "\gamma_Y"] \arrow[d, "{(\gamma_x, \gamma_y)}"] \arrow[dl, phantom, "H_x", bend left] \arrow[dr, phantom, "H_Y", bend right] & \\
X & X \times Y \arrow[l] \arrow[r] \arrow[u, phantom, "H"] & Y
\end{tikzcd}$$

$\boxed{\text{Products of groups \& products of spaces}}$

<u>Claim:</u> F is injective

<u>Pf:</u> claim that γ is a loop in $X \times Y$ that maps to the trivial element of $\pi_1(X) \times \pi_1(Y)$. We have a homotopy $H: I \times I \longrightarrow X \times Y$ between (γ_x, γ_y) and e ← identity.

construct homotopies

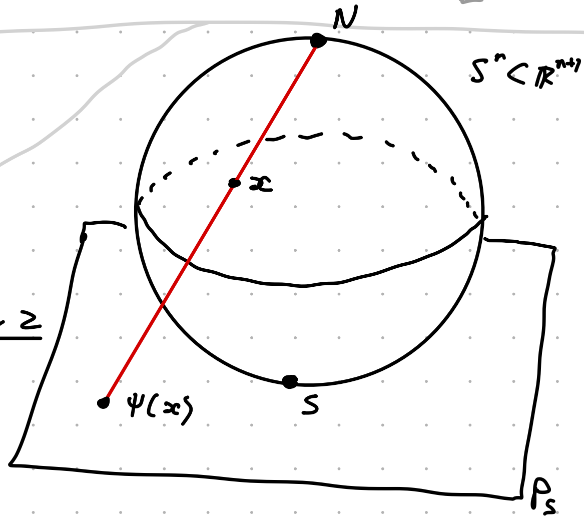
$H = (H_x, H_y)$ where H_x is a homotopy between γ_x and e

Two different things mapping to the same thing $\Rightarrow$ construct a homotopy to show they are the same. | and H_γ is a homotopy between γ_γ and e. ← the constant path...

Corollary: $\pi_1(T^2) = \pi^1(S_1) \times \pi^1(S_1) = \mathbb{Z} \times \mathbb{Z}$ ← torus...

Pf: $T^2 = S^1 \times S^1$ ← isomorphic

corollary: $\pi_1(T^n) = \mathbb{Z}^n$



The Fundamental Group of the n-sphere for $n \geq 2$

Stereographic Projection

Let $N = (0,0,\dots,0,1)$, $S = (0,0,\dots,0,-1)$

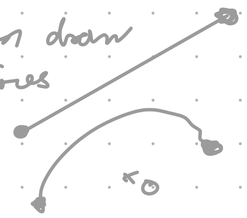
There is a homomorphism $\psi_N : S^n - \{N\} \longrightarrow P_S$

Also, we have $\psi_S : S^n - \{S\} \longrightarrow P_N = (\{x_1, x_2, \dots, x_n, 1\})$

Let $U_1 = S^n / \{N\}$, $U_2 = S^n / \{S\}$,

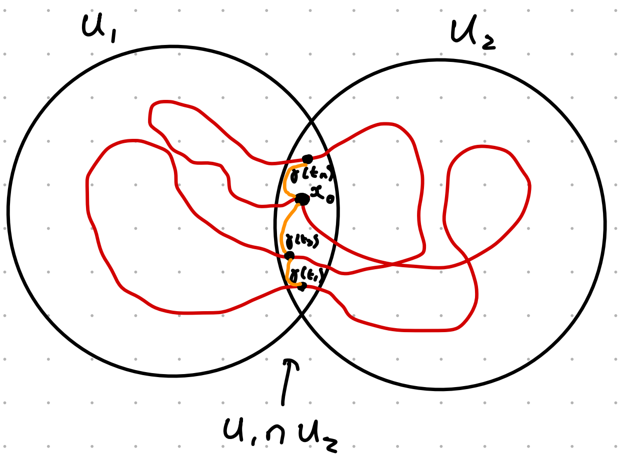
claim $U_1 \cap U_2 \cong \mathbb{R}^n / \{0\}$ is path connected ← why? can draw these lines

↑ homeomorphic



Proposition: For $n \geq 2$, $x_0 \in S^n$, $\pi_1(S^n, x_0)$ is trivial group.

Proof: Say we have a loop γ, assume $x_0 \neq N, S$, based at x_0

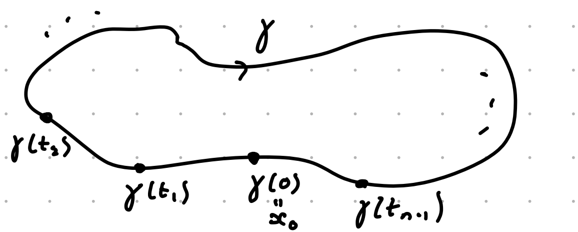


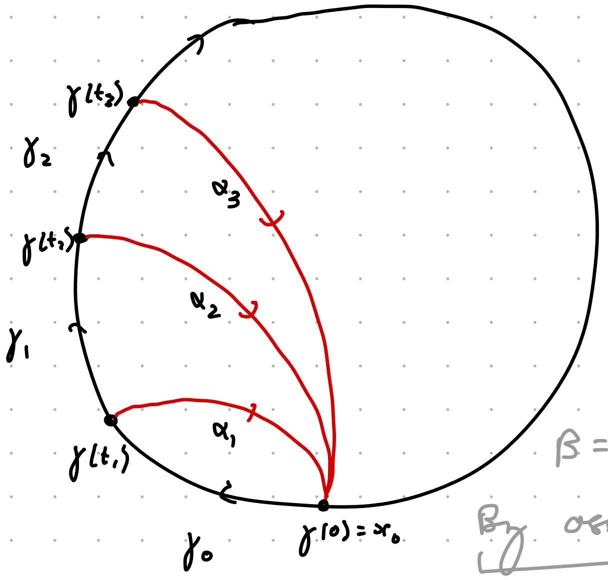
$\gamma : [0,1] \longrightarrow S^n$ is a loop

$\gamma : [0,1] \longrightarrow S^n$. ~~Consider $\gamma^{-1}(U_1)$ and $\gamma^{-1}(U_2)$~~

~~Those cover I, Find a finite subcover.~~ The $\alpha \subset U_1 \cap U_2$ ← lie completely in U_1 or completely in U_2

choose $t_0 = 0, \dots, t_n = 1$ s.t. $\gamma([t_j, t_{j+1}]) \subset U_1$ or U_2





WTS γ is trivial. How? separation

Let $\gamma_j(s) = \gamma(t_j + s(t_{j+1} - t_j))$

$$\gamma_j : [0,1] \longrightarrow S^2$$

let paths $\longrightarrow$ loops

$$\beta = (\gamma_0 \cdot \alpha_1) \cdot (\bar{\alpha}_1 \cdot \gamma_1 \cdot \alpha_2) \cdot \ldots \cdot (\bar{\alpha}_{m-1} \gamma_{m-1} \alpha_m) \cdot (\bar{\alpha}_m \cdot \gamma_m)$$

By assumption, $\gamma_1 \cdot \alpha_1 \simeq e_{x_0}$
$U_1 \cap U_2$ path connected $\quad \bar{\alpha}_1 \cdot \gamma_2 \cdot \alpha_2 \simeq e_{x_0}$ identity

$\therefore \quad \beta \simeq e_{x_0} \cdot \ldots \cdot e_{x_0} \simeq e_{x_0}$ rel ∂ homotopy of paths

Reassociating: $\beta = \gamma_1 \cdot \underbrace{(\alpha_1 \cdot \bar{\alpha}_1)}_{\simeq e_{\gamma(t_1)}} \cdot \gamma_2 \cdot \underbrace{(\alpha_2 \cdot \bar{\alpha}_2)}_{\simeq e_{\gamma(t_2)}} \cdot \ldots \cdot \underbrace{(\alpha_m \cdot \bar{\alpha}_m)}_{\simeq e_{\gamma(t_m)}} \cdot \gamma_m$

$\simeq \gamma_1 \cdot \ldots \cdot \gamma_m$ rel ∂

$\simeq \gamma$ rel ∂

conclude by transitivity of homotopies $\gamma \simeq e_{x_0}$ rel ∂ ∎

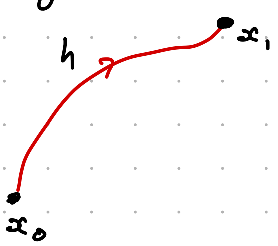
The technique for this proof can be used in other situations.

Say that $U_1 \cap U_2$ are path connected; U_1, U_2 are not simply connected. The proof shows that every element of $\pi_1(U_1 \cup U_2)$ is a product of elements of $\pi_1(U_1)$ and $\pi_1(U_2)$

Def: A path connected space X is simply connected if $\pi_1(X, x_0) = \{[e_{x_0}]\}$ $(= \{0\})$

Note: Recall that $\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$\beta_h(f) = [\bar{h} \cdot f \cdot h]$, so S^n $\forall n \geq 2$ $\mathbb{R}^n$ is simply connected.



Corollary: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$

The Model Result

As much as we can prove w/ $\pi_1(X, x_0)$!

Proof: Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is a homeomorphism. Then $f : \mathbb{R}^2 / \{0\} \longrightarrow \mathbb{R}^n / \{f(0)\}$ is also a homeo.

In case $n=1$: $\mathbb{R}^1/\{0\}$ disconnected, $\mathbb{R}^2/\{f(0)\}$ connected so not homomorphic. $\mathbb{R}^2 \not\cong \mathbb{R}^1$. Using connectivity...

For any n, $\mathbb{R}^n/\{0\} \cong \mathbb{R}\times S^{n-1}$ [homework problem] so

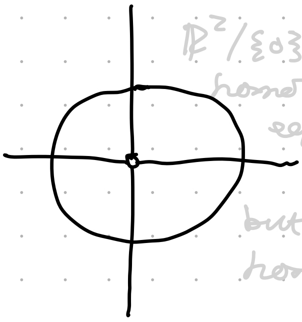
$$\begin{aligned}
\pi_1(\mathbb{R}^n - \{0\}) &= \pi_1(\mathbb{R}\times S^{n-1}) \\
&= \underbrace{\pi_1(\mathbb{R})}_{\leftarrow \text{trivial gp}} \times \pi_1(S^{n-1}) \\
&= \{0\} \times \pi_1(S^{n-1}) \\
&= \pi_1(S^{n-1})
\end{aligned}$$

Thus, $\pi_1(\mathbb{R}^2 - \{0\}) = \pi_1(S^1) = \mathbb{Z}$

For $n>2$, $\pi_1(\mathbb{R}^n - \{0\}) = \pi_1(S^{n-1}) = \{0\}$

Since homeomorphic spaces have the same fundamental group, π_1, see that $\mathbb{R}^2 - \{0\}$ not homeomorphic to $\mathbb{R}^n - \{0\}$ for $n > 2$

π_1 can be used to show that spaces are not homotopy equivalent to one another.

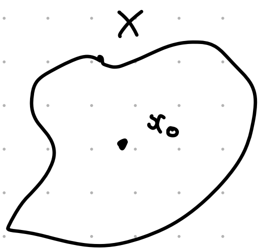


$\mathbb{R}^2/\{0\}$ is homotopy equivalent to S^1 but not homotopic.

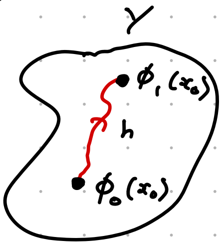
> **Prop:** If $\phi: X \longrightarrow Y$ is a homotopy equivalence, then $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$ is an isomorphism

Proof: Lemma 1st.

Lemma: If $\phi_t : X \longrightarrow Y$ homotopy, h a path from $\phi_t(x_0)$ formed by the image of the base point x_0 for $t\in[0,1]$, then



$h(t) = \phi_t(x_0)$



subscript is parameter from 0 to 1

$$\begin{tikzcd}
 & \pi_1(Y, \phi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & \\
 & \pi_1(X, \phi_0(x_0))
\end{tikzcd}$$

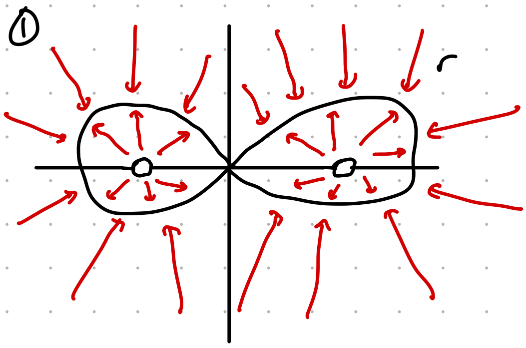
This diagram commutes

homotopies ignore base points

π_1 cares... connect!

Examples of Homotopy Equivalent Spaces

①



Deformation Retraction

$\mathbb{R}^2 / \{-1, 1\}$

A inclusion X

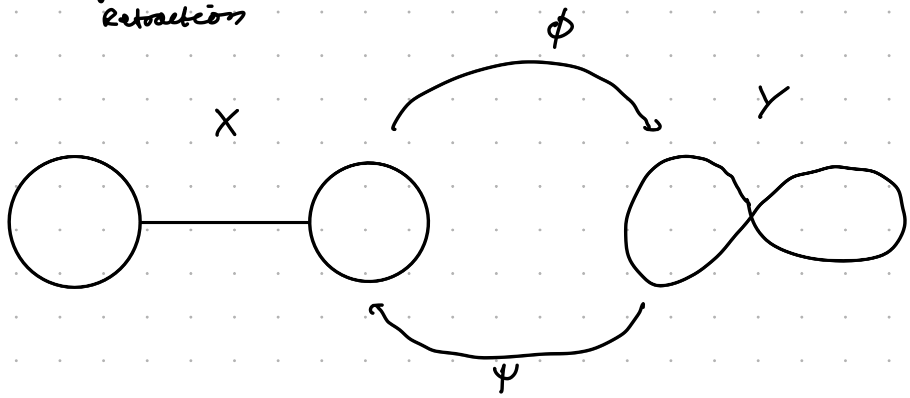
$$\begin{tikzcd} A \arrow[r, bend left, "i"] & X \arrow[l, bend left, "r"] \end{tikzcd}$$

retraction

$r \circ i = Id_A$

$i \circ r \simeq Id_X$ ← homotopic

②

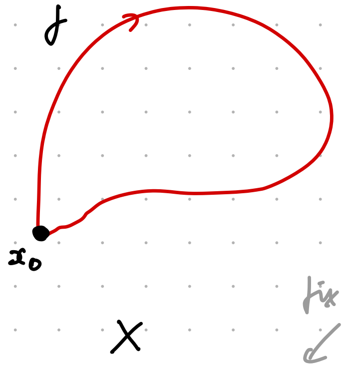


claim:

$\psi \circ \phi \simeq Id_X$

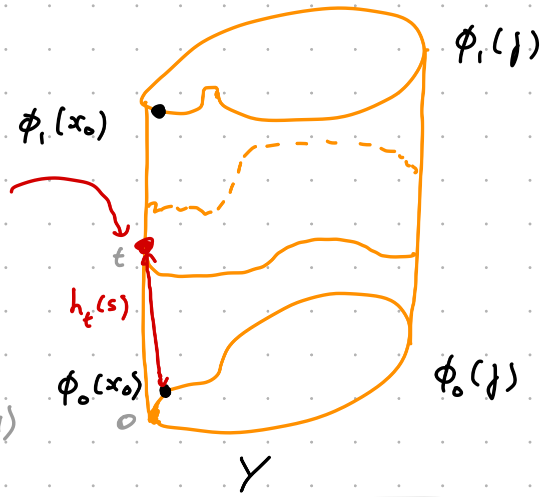
$\phi \circ \psi \simeq Id_Y$

Proof:
(of the lemma)



let

$h_t = \phi_t(x_0)$

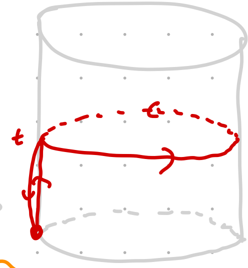


fix a particular t value

Define $h_t(s) = h(\underbrace{t \cdot s}_{\text{rescaling}})$ for $s \in (0,1)$

$s \in (0,1)$ so $ts \in (0,t)$

① Fix t



Define a homotopy, $\underbrace{h_t \cdot (\phi_t \circ f) \cdot \overline{h_t}}$

This is the path for some t fixed.

What is this?

$h_t \cdot (\phi_t \circ f) \cdot \overline{h_t}$

This shows that
$\phi_0(f) \simeq B_h(\phi_1(f))$
$\Rightarrow \underbrace{[\phi_0(f)] = [B_h(\phi_1(f))]}_{\text{This is the statement that the diagram commutes}}$

When $t = 0$, this is the path $\phi_0(f)$

When $t = 1$, this is the path $h \cdot (\phi_1(f)) \cdot \overline{h} = B_h(\phi_1(f))$
up, around & back again

(where $B_h(f) = [h \cdot f \cdot \overline{h}]$

To show diagram commutes, we have constructed a homotopy between these two loops
$\Rightarrow$ have the same homotopy class ■

Prop: If $\phi : X \longrightarrow Y$ is a homotopy equivalence, then $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$ is an isomorphism

$$\begin{tikzcd} X \arrow[loop left, "Id_X"] \arrow[r, bend left, "\psi"] & Y \arrow[loop right, "Id_Y"] \arrow[l, bend left, "\phi"] \end{tikzcd}$$

Proof:

$$\begin{tikzcd} & \pi_1(Y, \phi(x_0)) \arrow[dr, "\psi_*"] & \\ \pi_1(X, x_0) \arrow[ur, "\phi_*"] \arrow[rr, "(\psi\circ\phi)_*"] \arrow[dr, "Id_X"'] & & \pi_1(X, \psi\circ\phi(x_0)) \\ & \pi_1(X, x_0) \arrow[ur, "\beta_h"'] & \end{tikzcd}$$

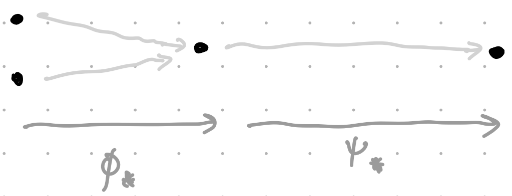
lemma says this diagram commutes

isomorphism too

This change of basepoint map is an isomorphism. why? can write down an inverse! $(\beta_h)^{-1} = \beta_{\bar{h}}$

Claim that $\psi_* \circ \phi_*$ is a bijection $\left[\begin{array}{c}\text{route on bottom} \\ = \\ \text{route on top}\end{array}\right]$

This shows that ϕ_* is injective.



If $\phi_*(x) = \phi_*(y)$ for $x \neq y$

then $\psi_*(\phi_*(x)) = \psi_*(\phi_*(y))$ but this is a bijection

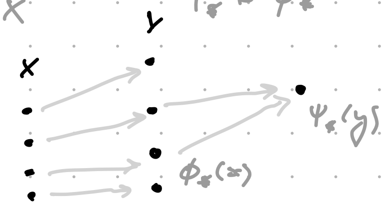
$\Rightarrow x = y$ ~~X~~ so injection

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\phi_*"] \arrow[rr, bend right, "\text{bijection so } \phi_* \text{ injective}"'] & \pi_1(Y, \phi(x_0)) \arrow[r, "\psi_*"] \arrow[rr, bend left, "\text{bijection so } \psi_* \text{ injective}"] & \pi_1(X, \psi_*\phi_*(x_0)) \arrow[r, "\phi_*"] & \pi_1(Y, \phi(\psi(\phi(x_0)))) \end{tikzcd}$$

Claim ϕ_* is a surjection.

Proof: Say $y \notin Im(\phi_*)$, then $\psi_*(y)$ is in the image of $\psi_* \circ \phi_*$

$\psi_*(y) = \psi_*(\phi_*(x))$ for some $x \in X$



This shows ψ_* not injective ~~X~~

$\Rightarrow$ so ϕ_* is a surjection

$\Rightarrow \phi_*$ is a bijection so isomorphism.

Def: A space X is contractable if it is homotopy equivalent to a point.

e.g. $\mathbb{R}^n$ is contractable (linear homotopy)

Corollary: A contractable space is simply connected.

Proof: Use lemma, $\pi_1(X, x_0) = \{0\}$ & isomorphism

Corollary: S^2 and T^2 are not homotopy equivalent

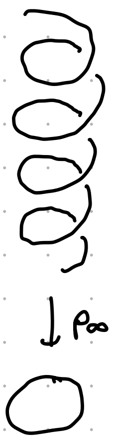
Proof: different π_1 $\pi_1(S^2) = \mathbb{Z}^2$, $\pi_1(T^2) = \{0\}$

Corollary: For $n > 2$, S^n & S^1 are not homotopy equivalent

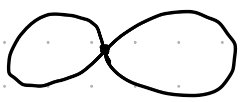
Proof: Different fundamental groups. $\mathbb{Z}^n$ and $\mathbb{Z}$ w/ $n \geq 2$

Note: S^n for $n \geq 2$ is simply connected but not contractable.
↳ Need more invariants for this.

In understanding S^1, the helix picture was important.

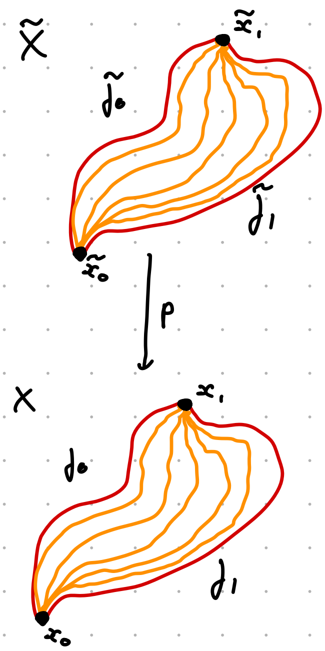


We want to develop a more robust theory of covering spaces, and their connections w/ fundamental groups.



What is the fundamental group of the figure 8?
Is there a nice covering space that plays the role of the helix?

<u>Homotopy Lifting Theorem (for homotopies rel ∂)</u>



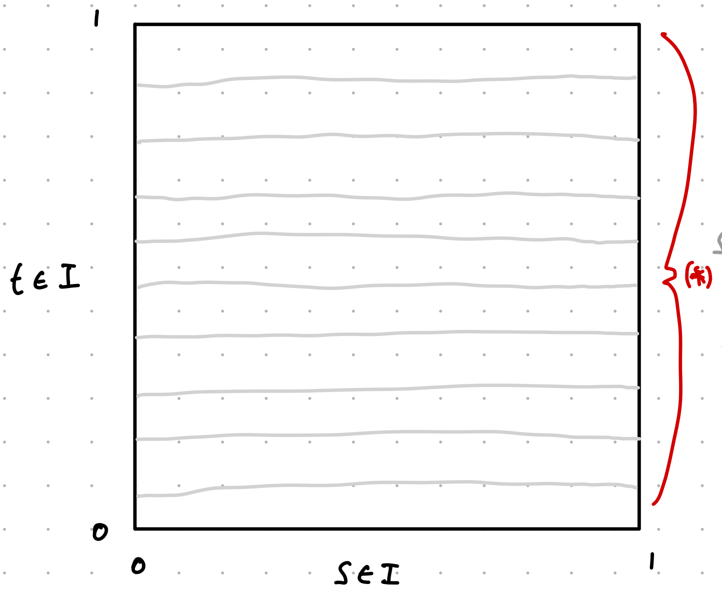
Let $p: \tilde{X} \longrightarrow X$ be a covering map

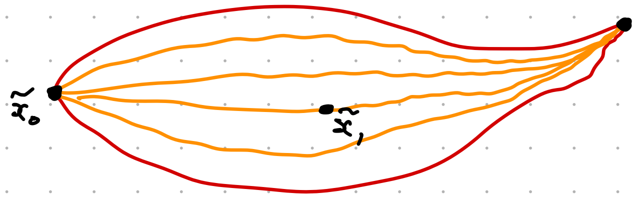
Given a homotopy rel ∂, $f_t : I \longrightarrow X$ w/
$f_t(0) = x_0$, $f_t(1) = x_1$ $\forall t \in I$

Given an $\tilde{x}_0 \in \tilde{X}$, there is a unique lift $\tilde{f}_t$ w/ $\tilde{f}_t(0) = \tilde{x}_0$

Set where $\tilde{x}_0$ lifts & the $\tilde{f}_t$ lift to the same endpoint.

Proof: Apply the general lifting thm w/ $Y = I$

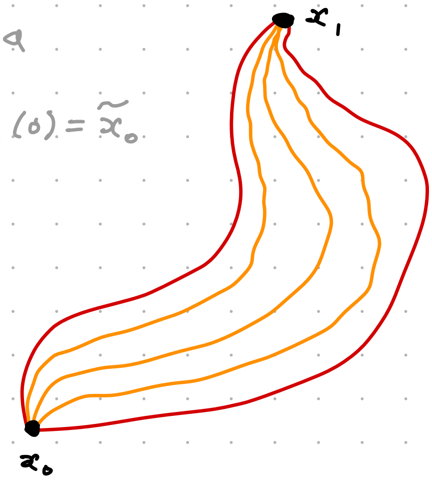




Specify that $\tilde{J}_t(x_0) = \tilde{x}_0$

GLT gives us a

$\tilde{J}_t(s)$ w/ $\tilde{J}_t(0) = \tilde{x}_0$



Apply uniqueness of lifts of paths to the path $t \longmapsto J_t(1)$ (*)

Note, $\tilde{x}_1 = \tilde{J}_0(1)$

Observe, the constant function $t \longmapsto \tilde{x}_1$ is a lift of $J_t(1)$ w/ $J_0(1) = \tilde{x}_1$

Two lifts of the same path w/ same initial point.

① $\tilde{J}_t(1)$ gives a lift of the constant path $J_t(1) = x_1$ with $J_0(1) = x_1$

② The constant path $t \longmapsto \tilde{x}_1$ gives a lift of the constant path $J_t(1) = x_1$

By uniqueness, $\tilde{J}_t(1) = x_1$

$\Longrightarrow$ conclude that the two paths are the same ∎

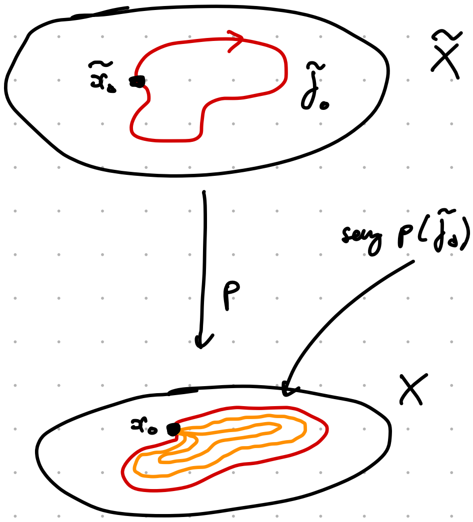
Say $p : \tilde{X} \longrightarrow X$ is a covering space.

Prop: The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by the covering map is injective and the image consists of homotopy classes of loops based at x_0, whose lifts are loops.

Example: $S^1 \xrightarrow{p_n} S^1$, $p_n(z) = z^n$

$\mathbb{Z} \xrightarrow{(p_n)_*} n\mathbb{Z}$ $\quad (p_n)_* : 0^e \longrightarrow 0 \in \mathbb{Z}$

Proof:



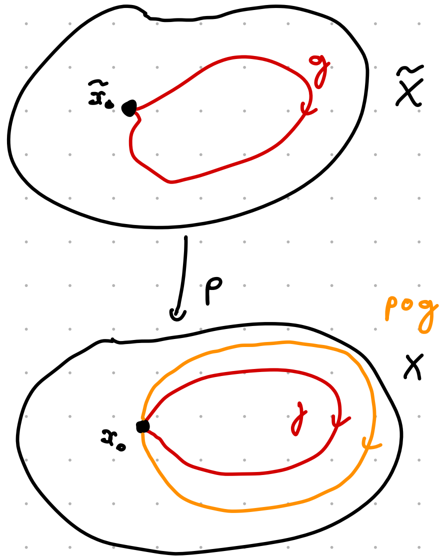
Say $\tilde{\gamma}_0 : I \longrightarrow \tilde{X}$ is a loop which is mapped to a trivial in X.

If $[p(\tilde{\gamma}_0)]$ is trivial, there is some homotopy γ_t with $\gamma_0 = \gamma_1$ and $\gamma_1 = e_{x_0}$

By prev. result, there is a lift $\tilde{\gamma}_t$ with $\tilde{\gamma}_0 = \tilde{\gamma}_0$ and $\tilde{\gamma}_1 = e_{\tilde{x}_1}$.

Thus, $\tilde{\gamma}_0$ is trivial in $\pi_1(\tilde{X}, \tilde{x}_0)$

A homotopy between $p \circ g$ and γ



① loop upstairs

② Projects down to a loop downstairs & γ is homotopic to this

↳ we don't know γ lifts to a loop.

③ If you lift you homotopy

$s=0$ $s=1$

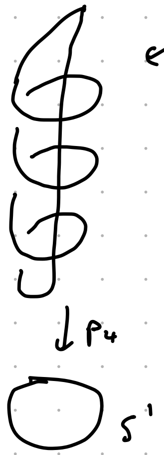
Fix $s \in \{0,1\}$ constant so they are loops ∵ constant ■

Monday 13th Nov 2023

Example covering space

① $p_4 : S^1 \longrightarrow S^1$ $\qquad p_4(z) = z^4$

glue top & bottom together



$(p_4)_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$

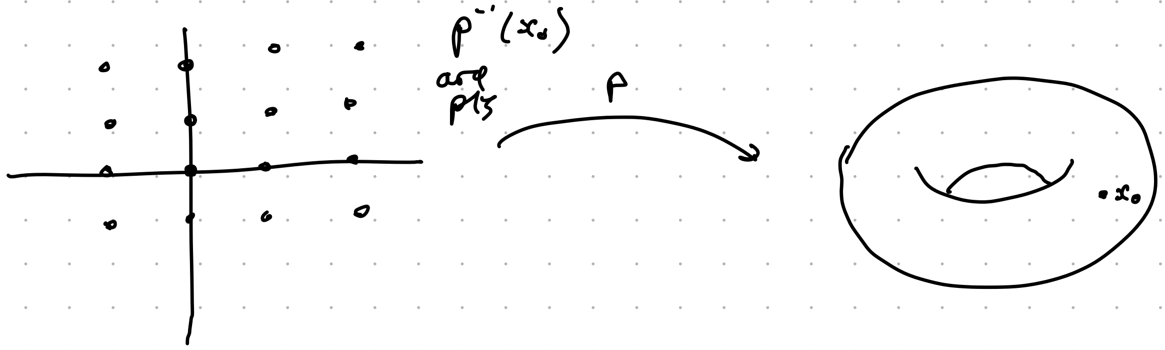
$\mathbb{Z} \longrightarrow \mathbb{Z}$

$(p_4)_*(n) = 4n$

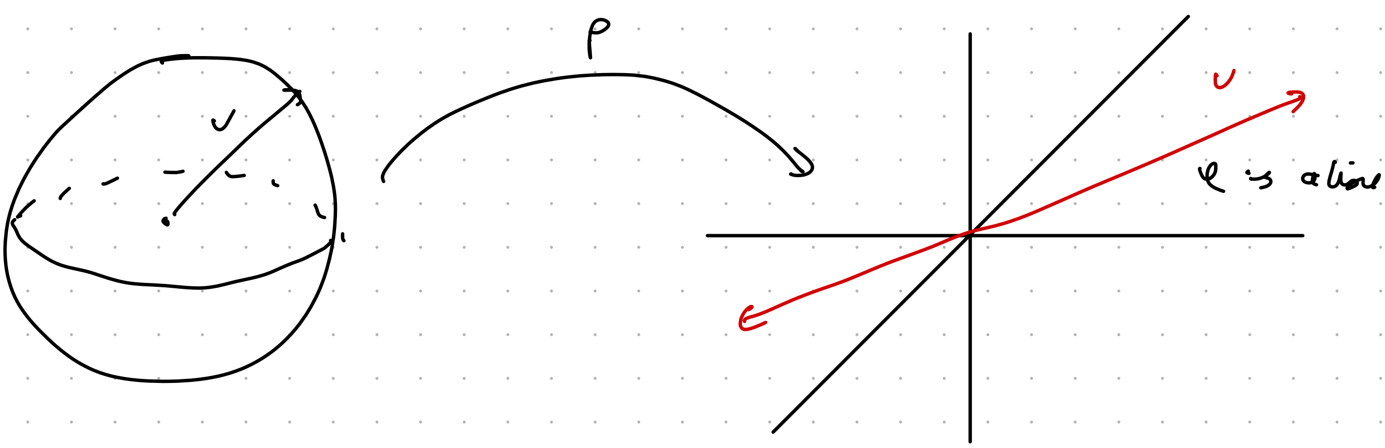
a closed loop in S^1 will map to a path under p_4^{-1}

② $p : \mathbb{R}^2 \longrightarrow T^2$

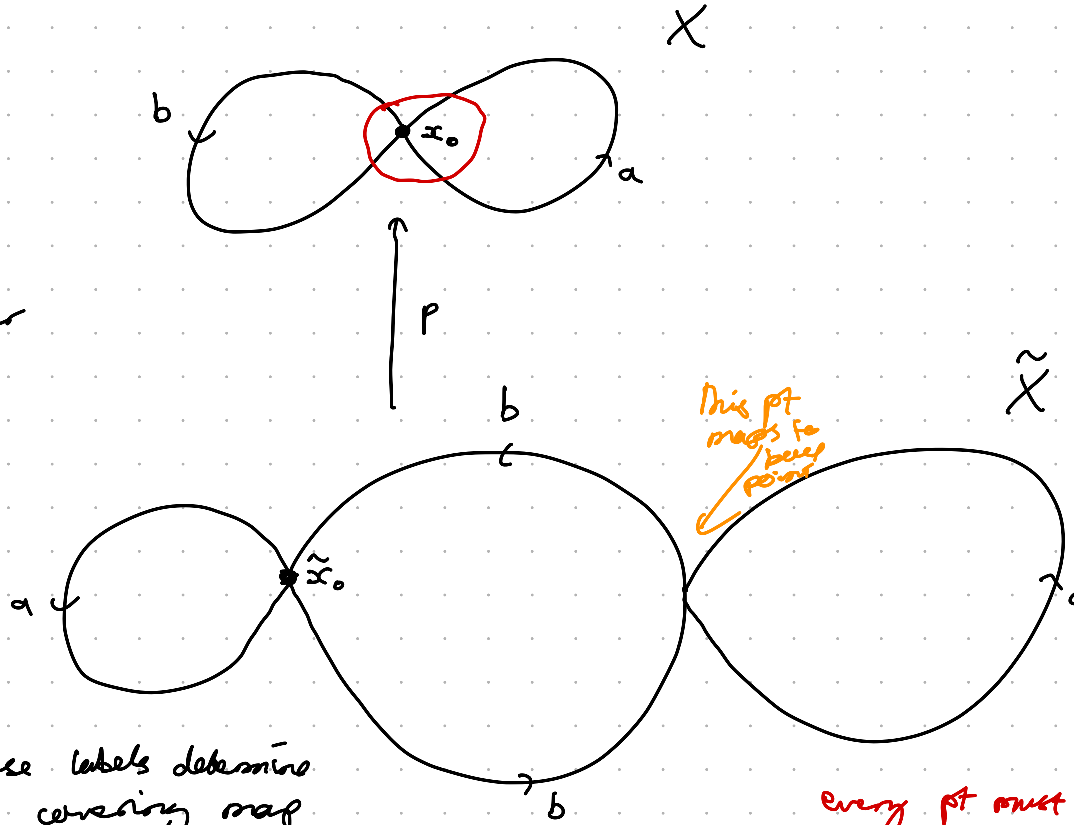
$p((x,y)) = (e^{2\pi i x}, e^{2\pi i y}) \in S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$



③ $p : S^2 \longrightarrow \mathbb{RP}^2$ ← set of lines in $\mathbb{R}^3$



④ Covering spaces of the figure 8



Consider

ⓐ

These labels determine a covering map

$$p:(\tilde{X},\tilde{x}_0)\longrightarrow (X,x_0)$$

every pt must be evenly covered

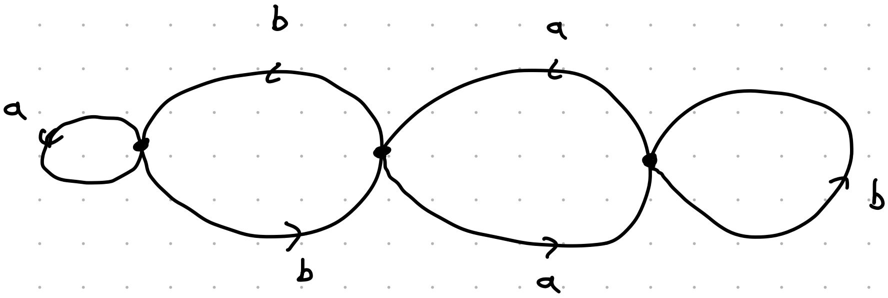
each $p^{-1}(U)$ with $x_0\in U$ should have 4 branches.

- a in, a out
- b in, b out

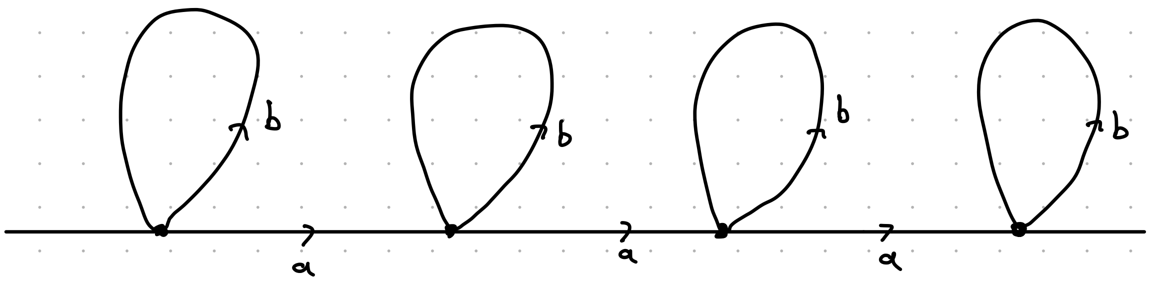
⇒ homeomorphism type explains the 4 products...

Lots of covering spaces of the figure 8. Lots of possibilities

ⓑ



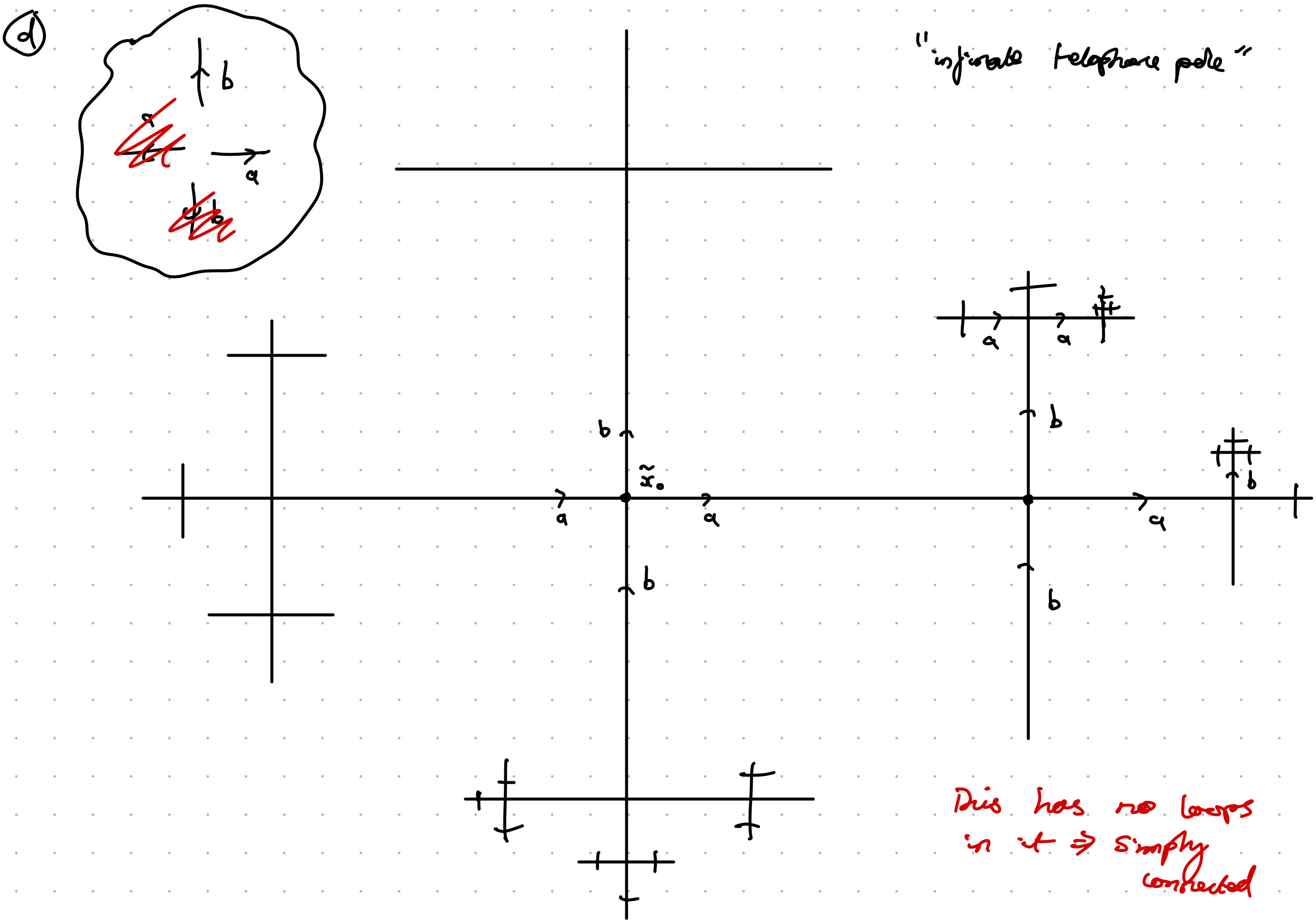
ⓒ



b in   b out
a in
a out

} each pt homeomorphic ... it works!

(d)



This has no loops in it $\Rightarrow$ simply connected

example of a simply connected covering space of the figure 8

$\hookrightarrow$ can mix finite & infinite to get

Given $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ a general covering map,
want to define a map

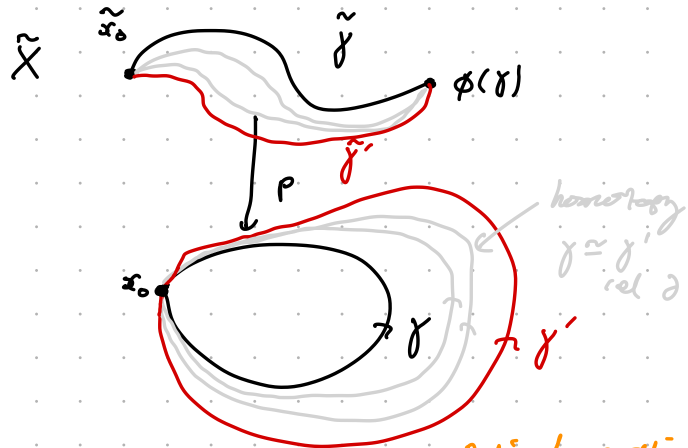
$$\phi: \{\text{loops at } x_0\} \longrightarrow p^{-1}(x_0) \subset \tilde{X}$$

say $\gamma: [0,1] \longrightarrow (X, x_0)$ loop.

$\because$ loop, there is a unique lift $\tilde{\gamma}: [0,1] \longrightarrow (\tilde{X}, \tilde{x}_0)$
s.t. $\tilde{\gamma}(0) = \tilde{x}_0$

$$p \circ \tilde{\gamma} = \gamma \quad \leftarrow \text{get } \gamma \text{ again}$$

project down



Define $\phi(\gamma) = \tilde{\gamma}(1)$

Note: If $\gamma' \simeq \gamma$ rel ∂,
Then if one do the lift

$$\tilde{\gamma}' \simeq \tilde{\gamma} \text{ rel } \partial$$

$\Rightarrow \tilde{\gamma}(0) = \tilde{\gamma}'(0)$ & $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$

In particular, $\phi(\gamma') = \tilde{\gamma}'(1) = \tilde{\gamma}(1) = \phi(\gamma)$

Defined operation on loops actually well defined on homotopy classes of loops!

Consider that ϕ is well defined on homotopy classes of loops, so have

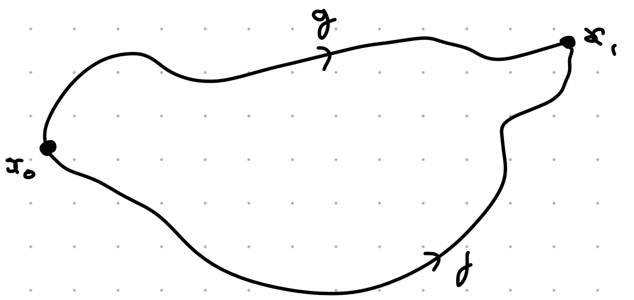
loop $\hookrightarrow$ lift it $\rightsquigarrow$ value at end pt?

$$\phi : \underbrace{\pi_1(X, x_0)}_{\text{group}} \longrightarrow \underbrace{p^{-1}(x_0)}_{\text{set}}$$

Thm: Given a covering $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$. If $\tilde{X}$ is path connected, then ϕ is surjective. If $\tilde{X}$ is simply connected, then ϕ is bijective.

[Helix is bijective, p_n are surjective]

Lemma: If $\tilde{X}$ is simply connected & f, g are paths in $\tilde{X}$ from x_1 to x_2. Then f, g are homotopic rel ∂.



Proof: Consider the path

$$f \cdot \bar{g} \cdot g$$

this first

$$\Rightarrow (f \cdot \bar{g}) \cdot g \overset{\partial}{\simeq} f \cdot (\bar{g} \cdot g)$$

$\underbrace{f \cdot \bar{g}}$ trivial by hypothesis

$$\overset{\partial}{\simeq} f \cdot e_{x_2}$$

$$\overset{\partial}{\simeq} e_{x_1} \cdot g \simeq f$$

$$\overset{\partial}{\simeq} g$$

$$\Rightarrow g \overset{\partial}{\simeq} f$$

Proof of Thm:

Case ①: Assume $\tilde{X}$ is path connected.

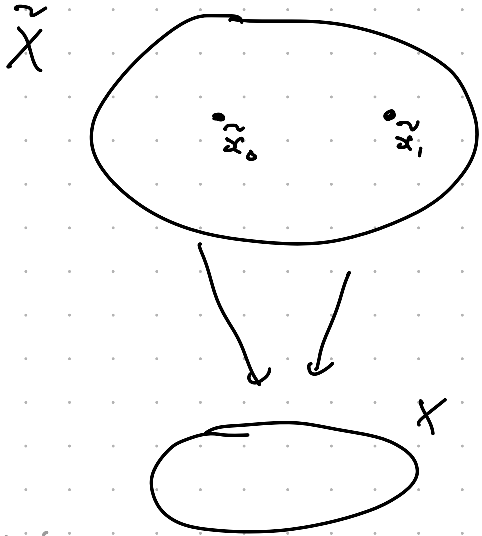
Say $p(\tilde{x}_0) = x_0$

let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$

let $p \circ \tilde{\gamma} = \gamma$

γ is a loop (w/ lift $\tilde{\gamma}$)

$\phi(\gamma) = \tilde{\gamma}(1) = \tilde{x}_1$ $\leftarrow$ proves surjectivity



Case ②: Assume $\tilde{X}$ is simply connected. WTS ϕ bijective

Say $\phi(\gamma) = \phi(\gamma')$

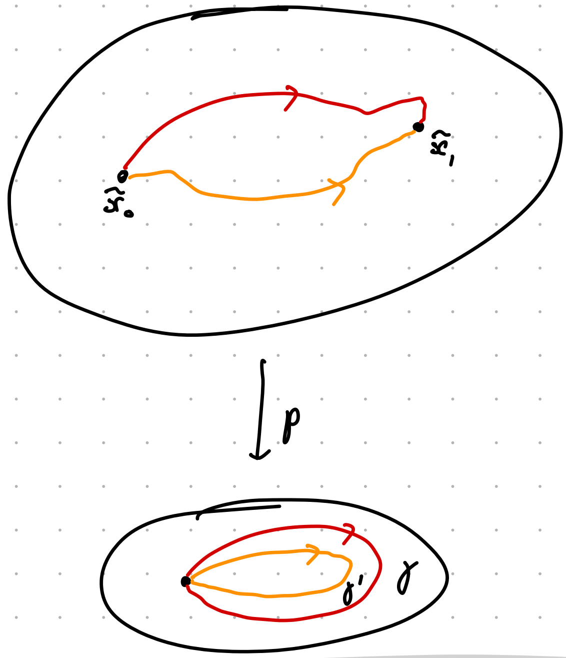
Since $\tilde{X}$ simply connected,

lemma $\Rightarrow$ $\tilde{\gamma} \overset{\partial}{\simeq} \tilde{\gamma}'$

Project the homotopy by p

$\Rightarrow$ $\gamma \overset{\partial}{\simeq} \gamma'$

$\Rightarrow$ $[\gamma] = [\gamma']$



Prop: $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

Proof: $p: S^2 \longrightarrow \mathbb{RP}^2$ is a covering of degree 2.

$\pi_1(S^2) = \{0\}$ $\because$ S^2 is simply connected.

We showed that in this case, ϕ is a bijection

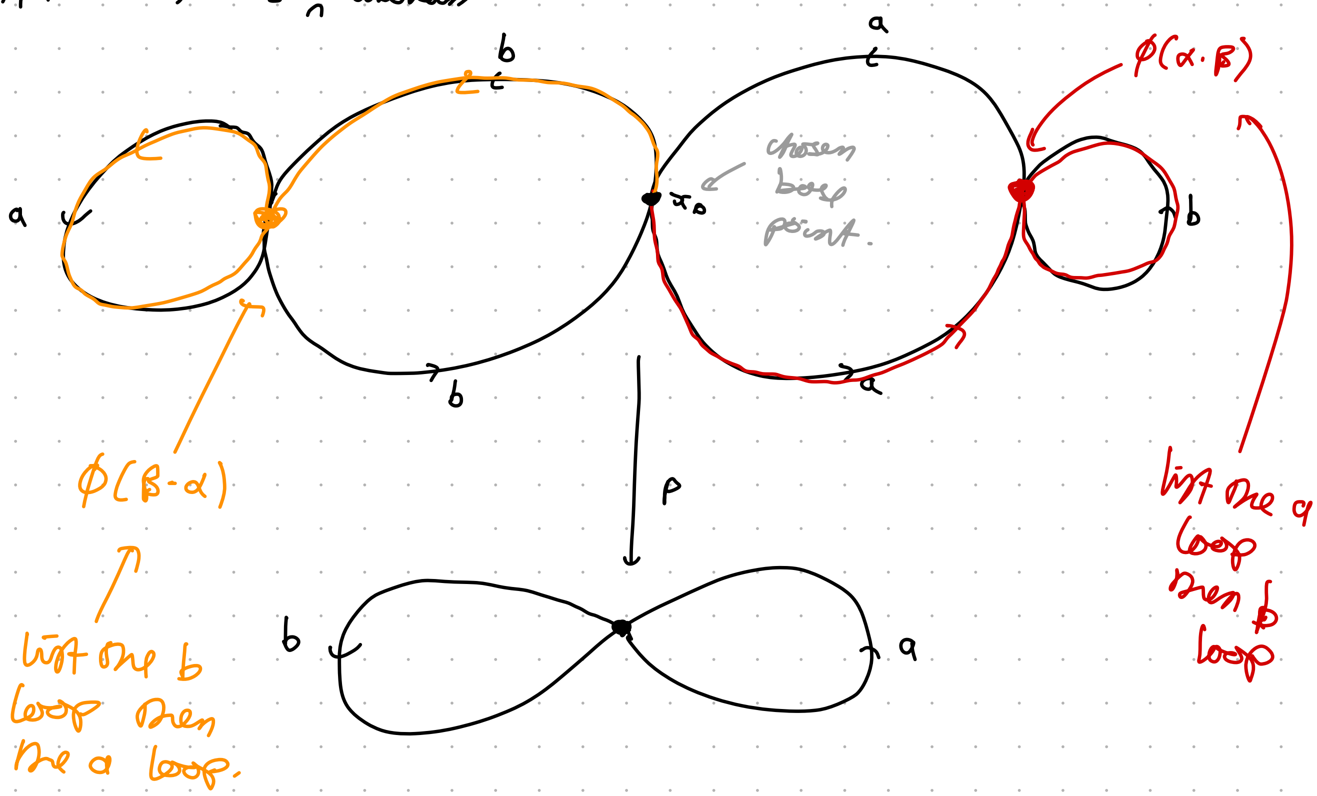
$\phi: \pi_1(\mathbb{RP}^2) \longrightarrow p^{-1}(x_0)$

We conclude that $\pi_1(\mathbb{RP}^2)$ has 2 elements,

There is a unique group w/ two elements, $\mathbb{Z}/2\mathbb{Z}$

Prop: $\pi_1(\infty, x_0)$ is non abelian

Proof:



Consider the loops $\alpha: [0,1] \to$  parametrising the right lob & $\beta: [0,1] \to$  parametrising the left lob.

WTS $[\alpha]\cdot[\beta] \neq [\beta]\cdot[\alpha]$ [elements in π_1 doesn't commute]

Since $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$ suffices to show

$$[\alpha\cdot\beta] \neq [\beta\cdot\alpha]$$

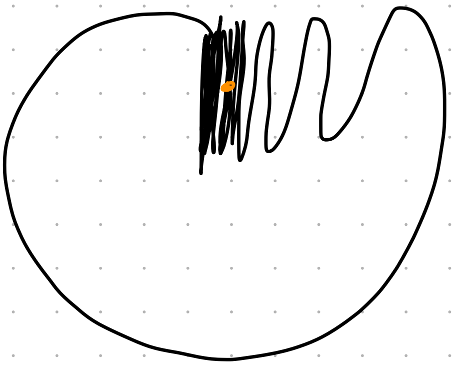
Compute $\phi(\alpha\cdot\beta)$ & $\phi(\beta\cdot\alpha)$

Conclude $\phi(\alpha\cdot\beta) \neq \phi(\beta\cdot\alpha)$ from computation.

But ϕ depends only on homotopy class rel ∂

$\Rightarrow [\alpha\cdot\beta] \neq [\beta\cdot\alpha]$ ∎

We say that a space (X, Ω) is $\boxed{\text{locally path connected}}$ if for any $x \in X$, and open set U containing x, then there is an open path connected set B with $x \in B \subset U$.



Topologist's sine curve.

path connected but not locally path connected.

This point •, a local neighbourhood of • will <u>not</u> be locally path connected.

This is equivalent to saying that the collection $\mathcal{B}$ of path connected open sets B is a basis for the topology.

$$U = \bigcup_{j\in J} B_j \qquad \{B_j\}_{j\in J} \quad B_j \in \mathcal{B}$$

<u>Lifting Criterion</u>

sometimes the lift exists!
sometimes it <u>does not</u> exist.

<u>Thm:</u> Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be covering space and $f: (Y, y_0) \to (X, x_0)$ a map w/ Y path connected & locally path connected.

Then a lift $\tilde{f}: (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

topology picture

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Take Y finite cover of circle
X circle
$\tilde{X}$ helix

no map from ~~...~~ finite cover to $\mathbb{R}$?

Q: maps of fundamental groups:

$$\begin{tikzcd} & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

algebraic question resolves the problem...

Proof: '$\Longrightarrow$'

If the lift $\tilde{f}$ exists,

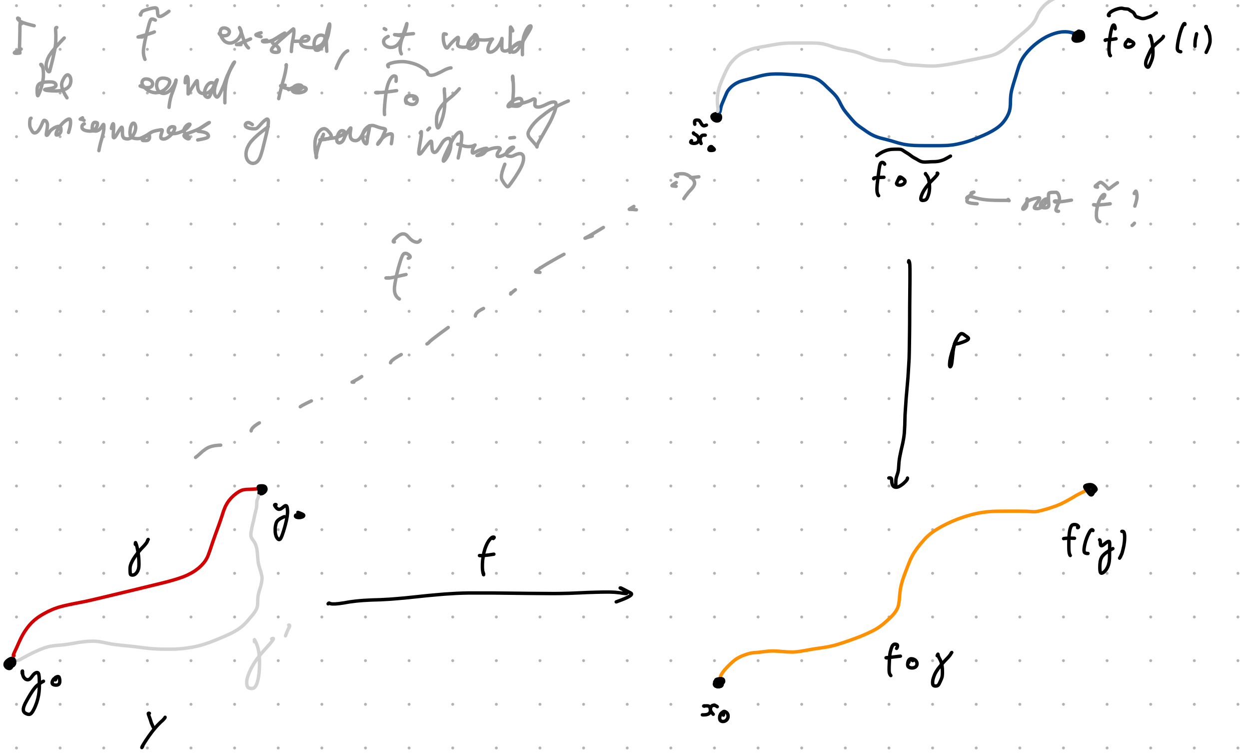
$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \supset p_*(\tilde{f}_*(\pi_1(Y, y_0))) = f_*(\pi_1(Y, y_0))$$

'$\Longleftarrow$' want to construct $\tilde{f}$ now.

Assume $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Let $y \in Y$, say γ path from y_0 to y

If $\tilde{f}$ existed, it would be equal to $\widetilde{f \circ \gamma}$ by uniqueness of path lifting



so $\tilde{f}(1)$ would be $\widetilde{f \circ \gamma}(1)$, in particular, we have uniqueness [same endpts]

$\Rightarrow$ just need to show existence now.

Define $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$.

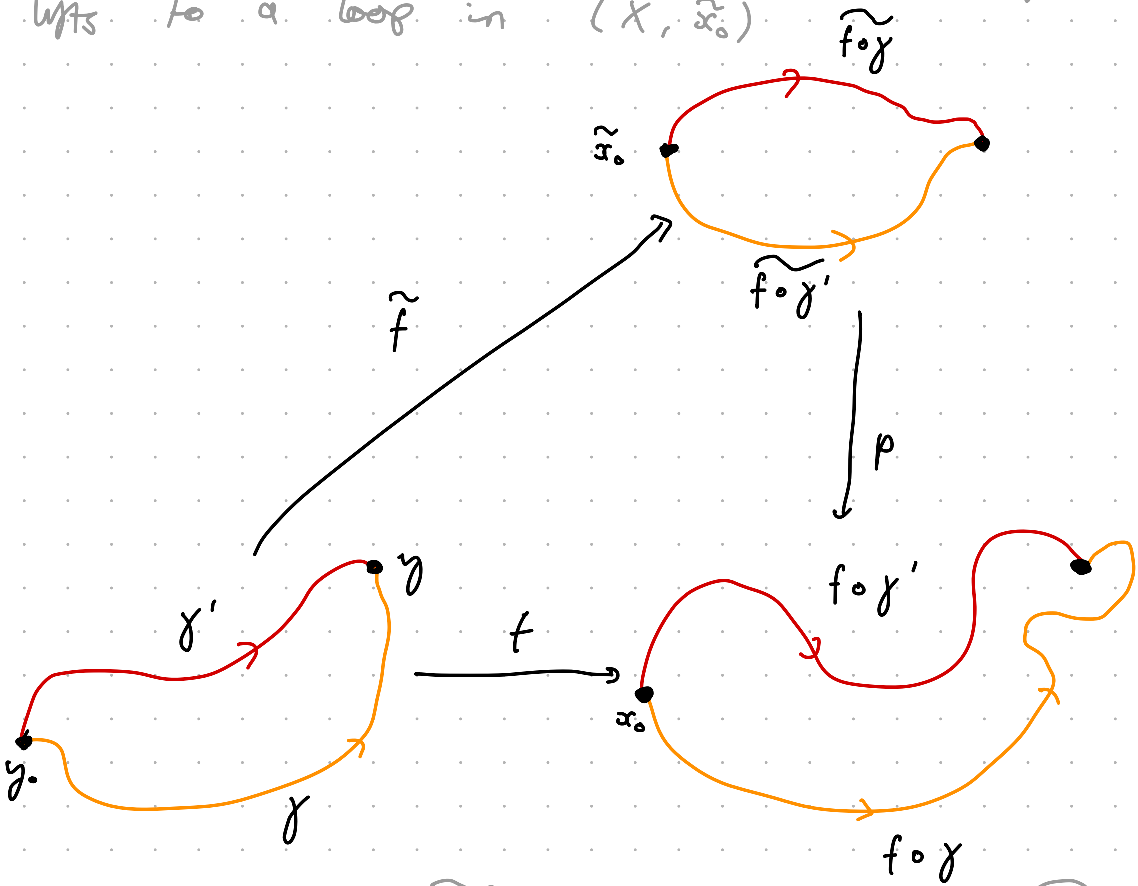
In order to make the definition, need to check that $\widetilde{f \circ \gamma}(1)$ is independent of γ ← just chosen to be some path...

Let γ' be a second path from y_0 to y.
WTS $\widetilde{f \circ \gamma'}$ doesn't lift somewhere else!

Consider the loop $\gamma' \cdot \bar{\gamma}$, its a loop so by hypothesis, the loop $(f \circ \gamma') \cdot (f \circ \bar{\gamma})$ is in the image of

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

From last week, we know $(f \circ \gamma') \cdot (f \circ \bar{\gamma})$ lifts to a loop in $(\tilde{X}, \tilde{x}_0)$



The endpoints ~~of~~ $\widetilde{f \circ \gamma}$ are the same as $\widetilde{f \circ \gamma'}$

$$\{\tilde{x}_0, \widetilde{f \circ \gamma}(1)\} = \{x_0, \widetilde{f \circ \gamma'}(1)\}$$

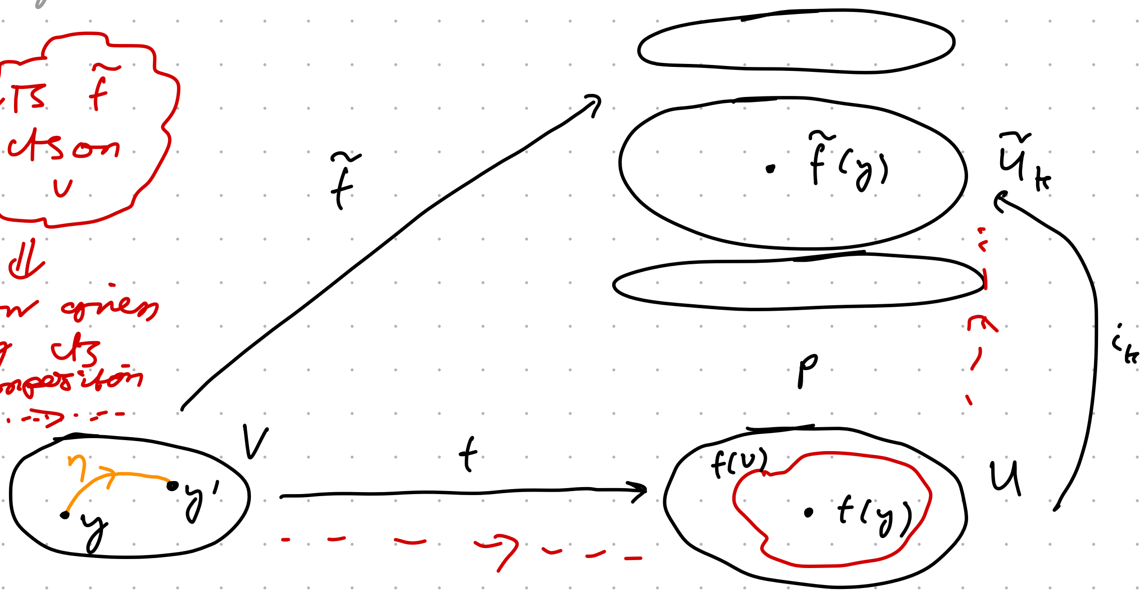
so $\widetilde{f \circ \gamma} = \widetilde{f \circ \gamma'}$ $\Rightarrow$ consistent way to define the lift $\tilde{f}$ that doesn't depend on the path.

Finally, need to show that $\tilde{f}$ is cts.

WTS $\tilde{f}$ cts on V

$\Downarrow$

show given by cts composition



Let U be a nbhd of $f(y)$ that is evenly covered. Let

$$p^{-1}(U) = \bigsqcup_k \tilde{U}_k$$

disjoint union

Property of covering space, i_k are local inverses to U.

Let V be a locally path connected nbhd of y so that $f(V) \subset U$

Pick $y' \in V$. Using path connectivity, there is a path η from y to y'.

Now, $i_k \circ f \circ \eta$ is a lift of $f(\eta)$ starting at $\tilde{f}(1)$.

Basically, obtain $\tilde{f}$ by f and then i_k.

$$\tilde{f}(y') = i_k(f(y')) \text{ \& } f \text{ is cts.}$$

def of $\tilde{f}$ is in terms of lifts. We use a particular lift

uniqueness of lifts says they're the same

so $\tilde{f} = i_k \circ f$

cts   cts

composition of cts functions

$\Rightarrow \tilde{f}$ cts.

Galois Theory of Covering Spaces

Part way through correspondence between spaces & groups.

Def: Let $p:(\tilde{X}, x_0) \longrightarrow (X, x_0)$ & $p':(\tilde{X}', \tilde{x}_0') \longrightarrow (X, x_0)$ be covering spaces. We say p & p' are equivalent if there is a homeomorphism $h:(\tilde{X}', \tilde{x}_0') \longrightarrow (\tilde{X}, \tilde{x}_0)$ so that

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, "h"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{x}_0') \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

Eg two covering spaces of the figure 8 are equivalent if there is a homeomorphism h between them preserving
- directions
- labels
- basepoint.

geometric idea

Prop: If X is path connected locally, then covering spaces, then coverering spaces $p:\tilde{X} \longrightarrow X$ and $p':\tilde{X}' \longrightarrow X$ w/ $\tilde{X}, \tilde{X}'$ path connected are equivalent iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ & $p_*(\pi_1(\tilde{X}', \tilde{x}_0'))$ are equal

algebraic.

Missed a whole week...

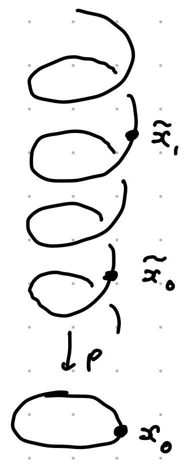
Universal covering space existence proof...

Proof that cover is simply connected...

Monday 27th November 2023

Covering space

$$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$$



Two covering spaces depend on base point.
What if you change base point.

Are these two covering spaces equivalent?

Yes, they are the same. It's a translation by 2. This is a deck transformation.

A deck group acts on the covering space.

In general...

Covering space $(\tilde{X}, \tilde{x}_0)$ & $(\tilde{X}, \tilde{x}_1)$ are equivalent

$$\Longleftrightarrow$$

There is a deck transformation $\tau : \tilde{X} \longrightarrow \tilde{X}$ taking $\tilde{x}_0$ to $\tilde{x}_1$

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[rr, "\tau"] \arrow[dr, "p"'] & & (\tilde{X}, \tilde{x}_1) \arrow[dl, "p"] \\ & (X, x_0) & \end{tikzcd}$$

<u>Assume:</u>
1. X connected
2. X locally connected
3. X locally simply connected

[using the machinery from last week]

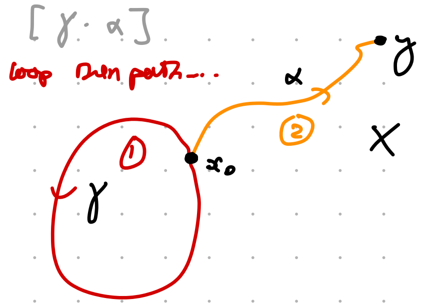
① <u>Prop</u> (for the universal cover $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$). The deck group is isomorphic to $\pi_1(X, x_0)$ & it acts transitively on $p^{-1}(x_0)$ [group action]

← one way to describe universal cover

"<u>Proof</u>": we know $\tilde{X} = \{[\alpha] : \alpha \text{ a path in } X \text{ starting at } x_0\}$

Let $[\gamma] \in \pi_1(X, x_0)$. Define $\tau_\gamma([\alpha]) = [\gamma \cdot \alpha]$

$$\boxed{\tau_\gamma \circ \tau_{\gamma'} = \tau_{\gamma \cdot \gamma'}}$$



② <u>Prop</u>: For every subgroup H, of $\pi_1(X, x_0)$, there is a covering space

$$(\tilde{X}_H, x_0) \quad \text{with} \quad \text{im}(p_*(\tilde{X}_H, \tilde{x}_0)) = H$$

"<u>Proof</u>": Start w/ universal cover $\tilde{X} = \{[\alpha] : \_\_\_\_\_\_\_\_\}$
Define equivalence relation on paths.

~~$\alpha \simeq \beta$ if $\alpha = h \cdot \beta$ for $h \in H$~~

$[\alpha] \sim [\beta]$ if $[\alpha] = [h \cdot \beta]$ for $[h] \in H$

Take $\tilde{X}_H = \tilde{X} / \sim$ [looking at the orbit space!]

(3)

Prop: Consider a covering space $p_H : (\tilde{X}_H, \tilde{x}_0) \longrightarrow (X, x_0)$

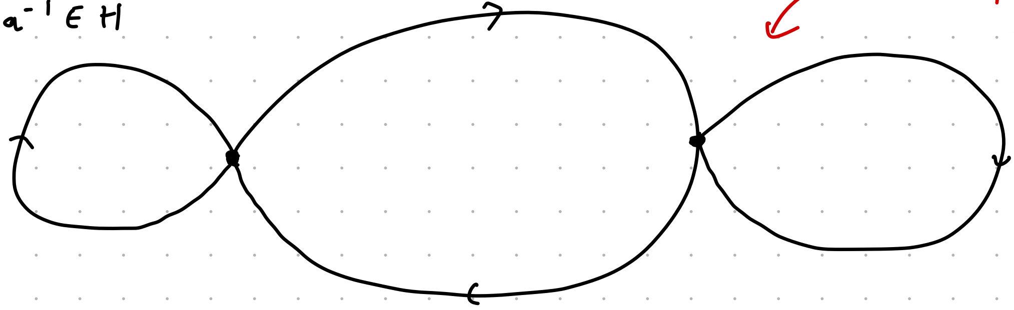
The deck group acts simply transitively on $p_H^{-1}(x_0)$ $\iff$ H is a normal subgroup of $\pi_1(X, x_0)$

Example: consider these covering spaces of ∞

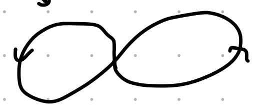
$b \in H_H$

$aba^{-1} \in H$

H_1

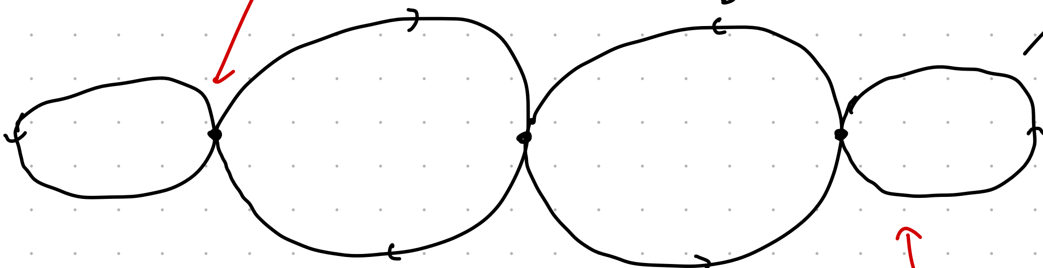


Deck group is transitive on vertices



This vertex can only go to itself.

H_2



$aba^{-1} \notin H_2$

$\Rightarrow$ H_2 is not normal.

Deck group is trivial.

Not acting transitively on vertices.

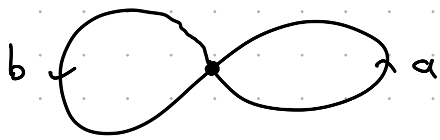
Not hard to prove, short course

Need to understand what they're saying...

What is $\pi_1(\infty, \cdot)$ $\boxed{\text{Free group}}$??

Not: $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}^2$,

<u>What is a free group?</u>



$\pi_1(\infty)$ is generated by 2 elements a & b.

Let G be some group generated by a and b. How distinguish $\pi_1(\infty)$ from G?

Say $G = \mathbb{Z}^2$ & $a = (1,0)$ $b = (0,1)$

Consider the word $a b \bar{a} \bar{b}$ This is equal to 1 in G [commute in G so can switch around]

Not equal to 1 in $\pi_1(\infty)$

$$a b \bar{a} \bar{b} = 1 \implies ab = ba \quad \times$$

$\mathbb{Z}^2$ abelian, $\pi(\infty)$ not abelian.

Let S be a set of generators of group G.

A word in S is some finite sequence of letters in $S \cup \bar{S}$ e.g. If $S = \{a, b\}$

word could be $a b b \bar{a} b a \bar{a} b$

A word w which corresponds to the identity 1 in G is a relation in G.

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$w = a b b \bar{a}$ formal word

$w_G = a b b \bar{a}$ evaluated in group G

$w = a \bar{a}$ non-formal word

$w_G = 1$ evaluates to identity in G.

Def: Given a word w in $S \cup \bar{S}$, we call a pair of letters $s \bar{s}$ in w an inverse pair. E.g.

$$w = a a b a \bar{b} b a \bar{a} a$$

Def: Two words w & w' are simply equivalent if one can be obtained from the other by inserting a single inverse pair.
w, w' equivalent if there is a chain of simple equivalences.

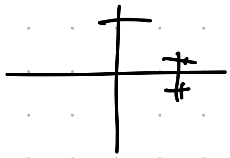
$$w \sim w_1 \sim \dots \sim w_n \sim w'$$

Def: w is reduced if there are no inverse pairs

Prop: Every word w is equivalent to a reduced word w_*

Proof: Induction on length. By cancelling pairs, we reduce the length. w equivalent to a word of minimal length. Possibly empty. $w = \emptyset$

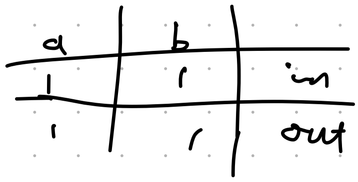
Now: want to build a copy of the universal cover of the figure 8.



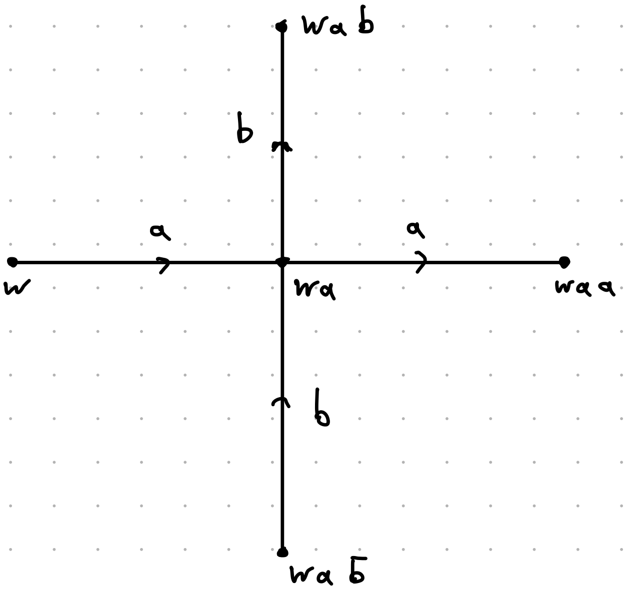
Let W be the words in $(a, b, \bar{a}, \bar{b})$ making a graph

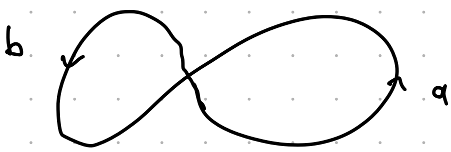
For each reduced word $w \in W$. Assign a vertex v_w.
For each reduced word $w \in W$, we attach an "a" directed edge

Attach "b" directed edges in the same way.



__claim:__ Covering of figure 8. WT check





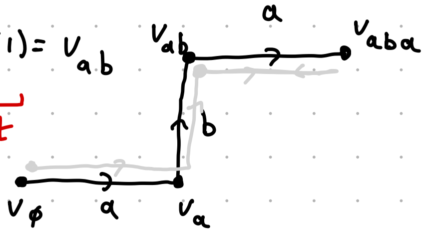
For any word w, reduced or not, there is a path γ_w in $^b\infty^a$, obtained by following the edges specified by the letters in the specified directions.

Let $\tilde{X}$ denote covering space. Let v_ϕ be the empty base point. $X = \infty$. Such a path γ in X has a unique lift to $\tilde{X}$ starting at v_ϕ

__Examples__

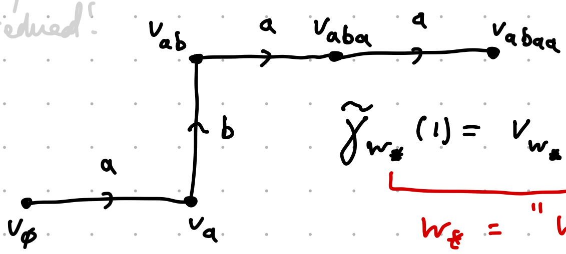
① $w = aba\bar{a}$

$\underbrace{\tilde{\gamma}_w(1)}_{\text{endpt}} = v_{ab}$



② $w_* = abaa$
reduced!

If w reduced $\Rightarrow$ path w/ no backtracking



$\tilde{\gamma}_{w_*}(1) = \underbrace{v_{w_*}}$ $(= v_{abaa})$

$w_* =$ "w reduced"

Prop 1: If w_* is reduced, $\tilde{\gamma}_{w_*}(1) = v_{w_*}$

Proof: Induction

figure ∞

corollary: If w_* is reduced but not empty, then $\gamma_{w_*} \in \pi_1(X, x_0)$ is **not** the identity.

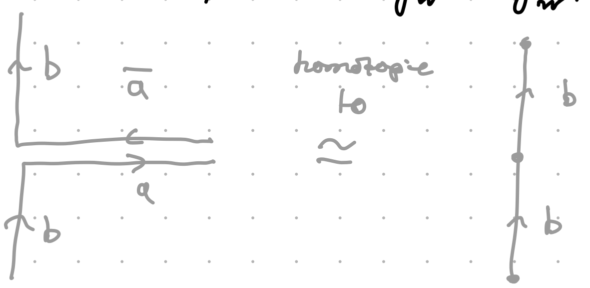
Proof: By homotopy lifting, $\tilde{\gamma}_{w_*}(1)$ is independent of the homotopy class of γ_{w_*}

If $\gamma_{w_*} \simeq$ constant, we will have $\tilde{\gamma}_{w_*}(1) = v_\phi$

Prop 2: If $w \sim w'$, then $\gamma_w \simeq \gamma_{w'}$

constructing elements in fundamental group of the cover of the figure 8

Proof:



If w is simply equiv to w'

If w is equiv to w', get a chain of homotopies.

$$\gamma_w \simeq \gamma_{w_1} \simeq \dots \simeq \gamma_{w_n} \simeq \gamma_{w'} \implies \gamma_w \simeq \gamma_{w'}$$

Prop 3: If $\gamma_w \simeq \gamma_{w'}$, then $w \sim w'$ [equivalent]

non-obv... depends where you cancel pairs...

Proof: Have ~~seen~~ seen w & w' each equiv. to reduced words w_* & w_*'

It follows that $\gamma_{w_*} \overset{\text{homotopic}}{\simeq} \gamma_w \overset{\text{assumption}}{\simeq} \gamma_{w'} \simeq \gamma_{w_*'}$

If we lift the loops. $\tilde{\gamma}_{w_*} \simeq \tilde{\gamma}_{w_*'}$ are homotopic rel ∂ as paths so

$$v_{w_*} = \tilde{\gamma}_{w_*}(1) = \tilde{\gamma}_{w_*'}(1) = v_{w_*'}$$

so

$$w \sim w_* = w_*' \sim w' \implies \underset{\text{equivalent}}{w \sim w'}$$

Made

Free Groups

28/11/23

Let $F(S)$ be a group where S is a set of generators for $F(S)$.

The Group $F(S)$ is a free group generated by S if for any group G & set map $f: S \longrightarrow G$, f extends uniquely to a homomorphism $\hat{f}: F \longrightarrow G$

$$\begin{tikzcd} F(S) \arrow[dr, "\hat{f}"] & \\ S \arrow[u, hook] \arrow[r] & G \end{tikzcd}$$

Examples

① $$\begin{tikzcd} F(a) \arrow[dr, dashed, "\hat{f}"] & \\ \{a\} \arrow[u, hook] \arrow[r, "f"] & G \end{tikzcd}$$

completely arbitrary

(free abelian groups if G restricted to abelian)

The infinite cyclic group generated by $\{a\}$ is the free group $F(a)$ $\{a^n : n \in \mathbb{Z}\}$

When is a word a relation in the group? Evaluated in G & Id_G.

One way of viewing free groups is they have the minimal # relations. Say $w \in S \cup \bar{S}$ a word. w gives an element w_F in the free group [evaluate in free group] $w = abba\bar{a}$. W also gives an element w_G in G if $w_F = 1$

$w_G = f(a)f(b)f(b)f(a)f(\bar{a}) \in G$

Then $\hat{f}(w_F) = \hat{f}(1) = 1$ so $w_G = 1$

$\hat{f}(w_F) = w_G$

Free groups are the minimal # of relations to make it a group

Only one identity

$\mathbb{Z}/n\mathbb{Z}$

$w_F =$

$ng = 1 \longrightarrow$ lots of relations

Prop: he changed in lecture...

Say that $F(S)$ & $F(S')$ are free groups, generated by sets with $\#S = \#S'$. A bijection $h: S \longrightarrow S'$ determines a unique isomorphism $f: F(\cancel{S}) \longrightarrow F(\cancel{S'})$ taking $s \in S$ to $h(s) \in S'$.

Proof:

$$\begin{tikzcd} F \arrow[dr, "\hat{f}"] & \\ S \arrow[u, hook] \arrow[r] & F' \end{tikzcd} \quad \& \quad \exists \hat{g} \text{ s.t. } \begin{tikzcd} F' \arrow[dr, "\hat{g}"] & \\ S' \arrow[u, hook] \arrow[r] & F \end{tikzcd}$$

$\hat{g}(s') = h^{-1}(s')$

WANT $\begin{tikzcd} F \arrow[r, bend left, "\hat{f}"] & F' \arrow[l, bend left, "\hat{g}"] \end{tikzcd}$

consider $\hat{g} \circ \hat{j}$

$\hat{g} \circ \hat{j}(s) = s$

uniqueness of homo $\Rightarrow$ $\hat{g} \circ \hat{f} = Id_F$

& $f \circ g = Id_{F'}$

Formal definitions

also . . .

WTF is going on!

Conclude: Any two free groups (generated by two elements a & b) are isomorphic.

we've described its properties & they are consistent. But, does it even exist???

A free abelian group does exist.

Does the free group $F(a,b)$ exist? Is there a group that satisfies these properties?!?

Prop: $\pi_1(\infty, x_0)$ is the free group generated by $\{a, b\}$.

we've invented this abstract definition that exactly describes this group!

Proof: Say we have a group G & a pair of elements, $\alpha, \beta \in G$. We need to show that there is a unique homomorphism $\hat{f} : \pi_1(\infty) \longrightarrow G$ with $\hat{f}(a) = \alpha$, $\hat{f}(b) = \beta$. [a, b are two loops]

if $w = a\,b\,a\,b\,b$

Given $\omega \in \pi_1(\infty)$, we can write ω as some word W in a & b. $\omega = W_{\pi_1(\infty)}$ [evaluated in the group]

$h(\omega) = W_G = \underbrace{\alpha\beta\alpha\beta\beta}_{\text{evaluated in } G}$

Define $h(\omega) = W_G$

What if we chose a different w' w/ $\omega = w'_{\pi_1(\infty)}$

WTS ω doesn't depend on word choice.

If $\omega \in \pi_1(\infty)$ is represented by two words w & w'.
& $\omega \simeq \gamma_w$ & $\omega \simeq \gamma_{w'}$, $\boxed{\text{PROP 3}} \Rightarrow \underbrace{w \sim w'}_{\text{is equivalent to}}$ so
$w \sim w_1 \sim \dots \sim w_n \sim w'$.

But, if $w_j \sim w_{j+1}$ is a simple equivalence, then

$\underbrace{(w_j)_G = (w_{j+1})_G}_{\text{when you evaluate the word, the inverse pair evaluates trivially.}}$

$w_j = \alpha\beta\alpha\bar{\alpha}\beta$

$w_{j+1} = \alpha\beta\cancel{\alpha\bar{\alpha}}\beta$

$\hat{f}$ def is well defined

In general $\hat{f}$ is well defined & uniquely determined.
word $(a, b) \longrightarrow \alpha\beta$ same.

$\hat{f}$ is a homomorphism $\because$ you stick words together & you get composition.

What have we gained? $\pi_1(\infty)$ is the free group on two generators.

$\pi_1(\infty)$ is reduced words. Take a word, reduce & it is unique.

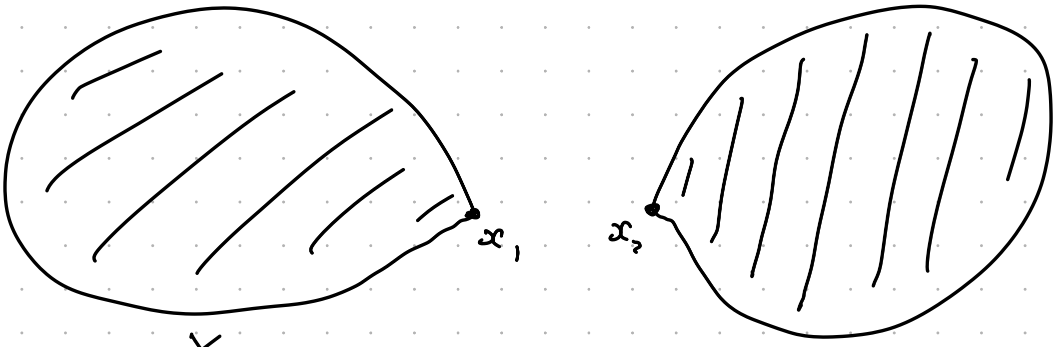
Fundamental group of figure ∞ & algebra Q of free group of 2 generators looks like reduced words.

Formal def $\longrightarrow$ positive outcome!

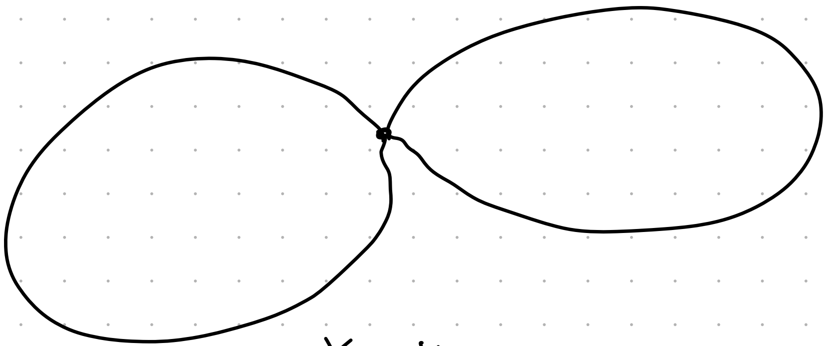
4/12/23

Say that (X_1, x_1) & (X_2, x_2) are pointed topological spaces.

Define $X_1 \vee X_2$ to be $X_1 \sqcup X_2 / x_1 \sim x_2$ w/ basepoint $x_1 = x_2$



X_1 X_2



$X_1 \vee X_2$

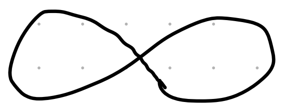
Example

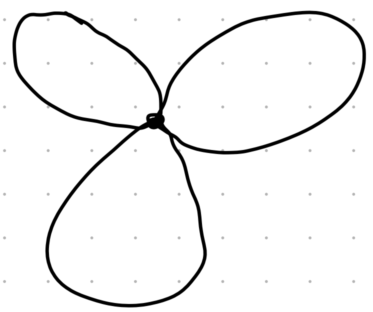
$S^1 \vee S^1 = \infty$

If we have a collection of spaces indexed by α

$$\bigvee X_\alpha = \bigsqcup_\alpha X_\alpha \Big/ \sim \quad \text{where } x_\alpha \sim x_{\alpha'} \quad \forall \alpha, \alpha'$$

The wedge of n circles is the rose w/ n petals





...

The fundamental group of the rose w/ n petals is the free group on n generators.

What about the rose w/ infinitely many petals?

Random shit about weak topology. CW complex...

Let G be a group w/ generating set S
[$\forall g \in G$ $g = s_1 s_2 s_3 \ldots, s_n$ (a word in S)]

$F(S)$

set of all reduced words on elements of S
Free group

$$\begin{tikzcd} F(S) \arrow[d] \arrow[dr, "\hat{f}"] & \\ S \arrow[r, hook, "f"] & G \end{tikzcd}$$

homomorphism
word in generators evaluated in G

By properties of free groups, there is a homomorphism $\hat{f}$ where $\hat{f}(w) = w_G$

reduced word

evaluated in G

$w = g_1^{n_1} \cdots g_m^{n_m}$

$w_G = $ multiplied out

Since S generates G, $\hat{f}$ is surjective

$$N \hookrightarrow F(S) \xrightarrow{\hat{f}} G$$

$N = \ker(\hat{f})$

N is a normal subgroup.

The elements of N are relations in G that map to the identity. N is called the relation subgroup.

The 1st isomorphism thm $\Rightarrow$ G is isomorphic to $F(S)/N$

$$G \cong F(S)/N.$$

So describing N tells us what G is.

How to describe N?

A set $R \subset N$ <u>normally generates</u> N if the elements of R & all their conjugates generate N.

R is called a complete set of relations.

G is determined by a set of generators $s_1, \dots, s_n$ & a complete set of relations $r_1, \dots, r_n$.

We write

$$G = \langle \underbrace{s_1, s_2, \dots, s_n}_{\text{generators}} \mid \underbrace{r_1, \dots, r_m}_{\text{relations}} \rangle$$

Examples

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$$

$$\mathbb{Z} = \langle a \rangle$$

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

$$\text{Dihedral} = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$$

$$= \langle a, b \mid a^2 = 1,\ b^n = 1,\ aba^{-1} = b^{-1} \rangle$$

2nd last lecture

5/12/23

In this setting, there is a covering space $\tilde{X}_N$ of the figure 8 $S^1 \vee S^1$

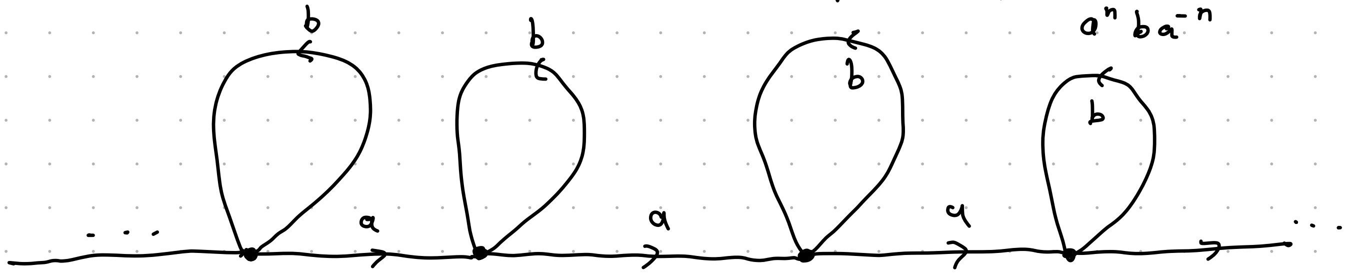
$$N \longrightarrow F(a,b) \longrightarrow G$$

$$F(a,b)/N \cong G$$

Example 1

$$G = \langle a, b \mid b = 1 \rangle$$

What does $\tilde{X}_N$ look like? N contains all conjugates of b
in particular, aba^{-1}
$a^n b a^{-n}$

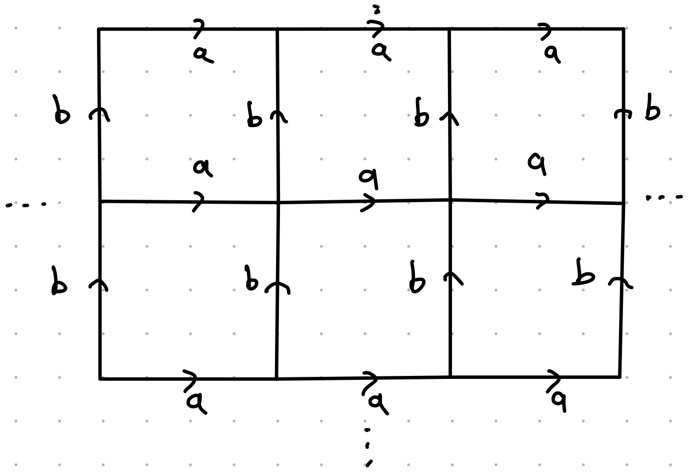


Deck group acts transitively on vertices. Any vertex can be taken to any other vertex.
Deck group on covering spaces.

Example 2

$(aba^{-1}b^{-1} = 1)$

$$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$$



$N = \mathbb{Z}^2$
deck group is translations

Van Kampen's Thm tells us how to describe the fundamental group of a space in terms of pieces of the space.

Case 1: $X = U_1 \cup U_2$ where U_1, U_2 path connected.
and $U_1 \cap U_2$ is simply connected.
Then $\pi_1(X) = \pi_1(U_1) * \pi_1(U_2)$

Free product of Groups

say $F = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

$F' = \langle g_1', \dots, g_{n'}' \mid r_1', \dots, r_{m'}' \rangle$

Define $F * F' = \langle g_1, \dots, g_n, g_1', \dots, g_{n'}' \mid r_1, \dots, r_m, r_1', \dots, r_{m'}' \rangle$

Examples $\mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle = \langle a, b \rangle$

$\{1\} = \{1\} * \{1\}$ $\quad\quad \langle \ \rangle = \langle \ \rangle * \langle \ \rangle$

Example $\pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$

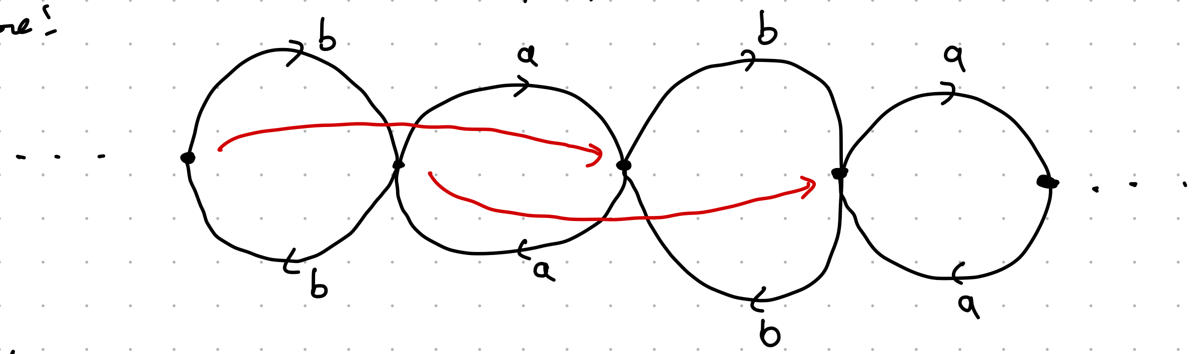
$\pi(S^n) = \{1\} \quad n \geq 3$

$\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

What does the caley graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ look like?

$\langle a, b \mid a^2 = b^2 = 1 \rangle$

Done!



Deck group is any map that preserves lables & directions

so $N = 2\mathbb{Z}$

Deckgroup contains ab, $ba = (ab)^{-1}$

Caley Graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ [Hatcher p.78]

Groups G, G', H & homeomorphism $f: H \longrightarrow G$, $f': H \longrightarrow G'$

say $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$, $G' = \langle g_1', \dots, g_{n'}' \mid r_1', \dots, r_{m'}' \rangle$

& H has generators $h_1, \dots, h_\ell$ Then

$G *_H G' = \langle g_1, \dots, g_n, g_1', \dots, g_{n'}' \mid r_1, \dots, r_m, r_1', \dots, r_{m'}', f(h_1) = f'(h_1), \dots, f(h_\ell) = f'(h_\ell) \rangle$

Note, $G *_H G'$ depend on f & f' even though they don't appear

Van Kampen Thm

If $X = U_1 \cup U_2$, U_1, U_2 open in X, $U_1 \cap U_2$ open in X, $U_1, U_2, U_1 \cap U_2$ path connected, then

$$\pi_1(X) = \pi_1(U_1) \underset{\pi_1(U_1 \cap U_2)}{*} \pi_1(U_2)$$

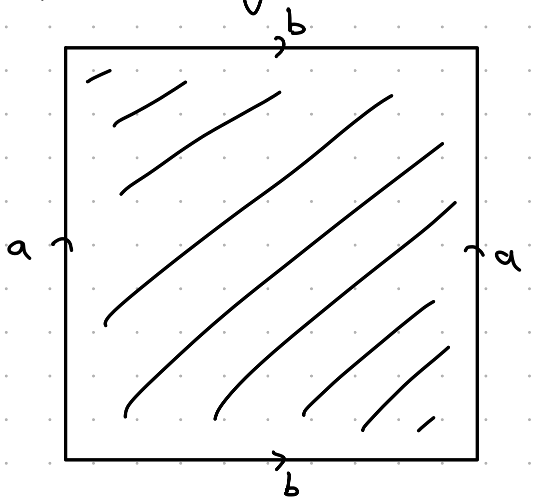
maps given by

$$\pi_1(U_1) \xleftarrow{(i_1)_*} \pi_1(U_1 \cap U_2) \xrightarrow{(i_2)_*} \pi_1(U_2)$$

(lecture 25)

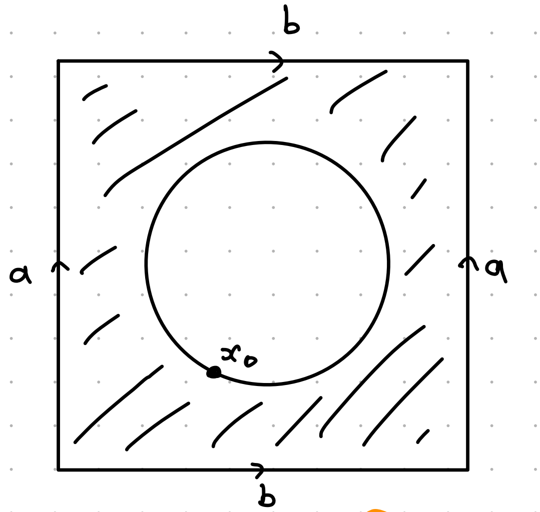
Application of Van Kampen to the Torus

5/12/23

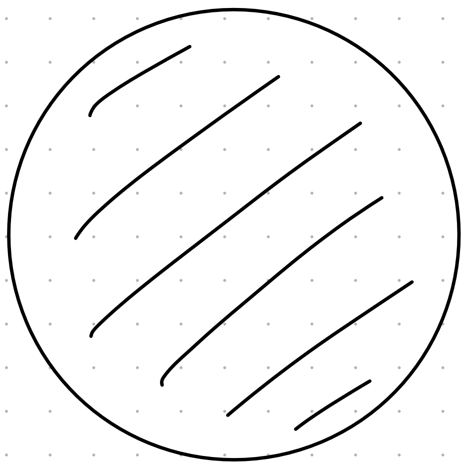


$= T^2$

Remove a disk



$= A$

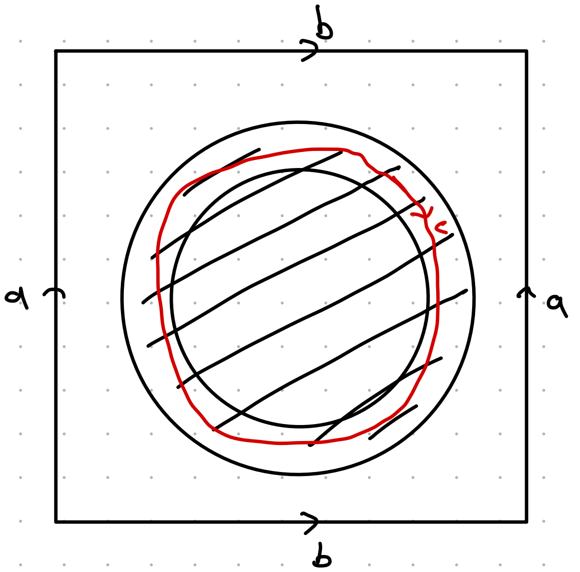


$= B$

larger disk

homotopic to figure 8 b ∞ a

$A \cup B$



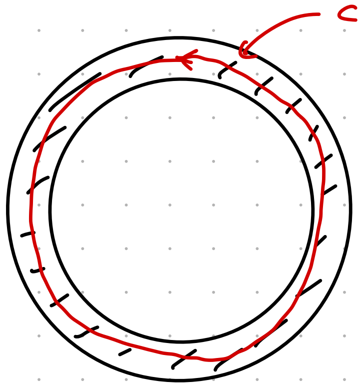
$= B$

$A \cup B$ covers the torus again

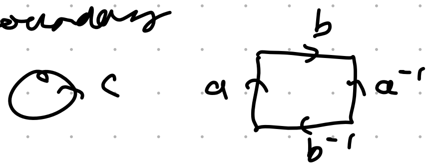
$T^2 = A \cup B$

$x_0 \in A \cap B$

$A \cap B$



c loop moves to boundary



$$\pi_1(A) \longleftarrow \pi_1(A \cap B) \longrightarrow \pi_1(B)$$

$$\pi_1(\infty) \xleftarrow{f_*} \pi(S^1) \xrightarrow{f_*'} \pi_1(D^2)$$

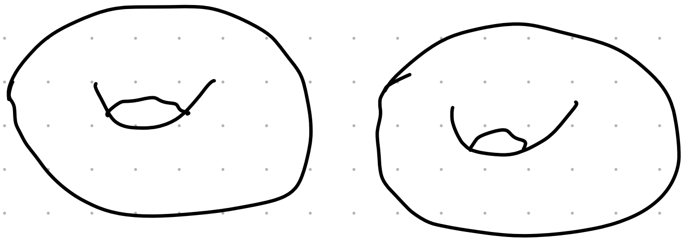
$\pi_1(\infty)$: homotopic to ∞; $\pi(S^1)$: circle; $\pi_1(D^2)$: disk

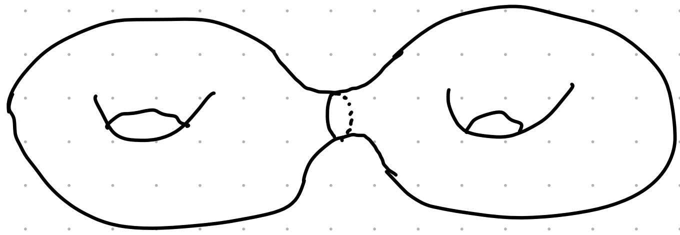
$$\underbrace{a, b \xleftarrow{aba^{-1}b^{-1}} c \longmapsto \{1\}}_{\text{generators, no relations}}$$

$$\underbrace{\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)}_{\text{amalgimated}} = \langle a, b \mid f_*(c) = f_*'(c) \rangle$$

$$= \underbrace{\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle}_{\text{presentation for the torus.}}$$

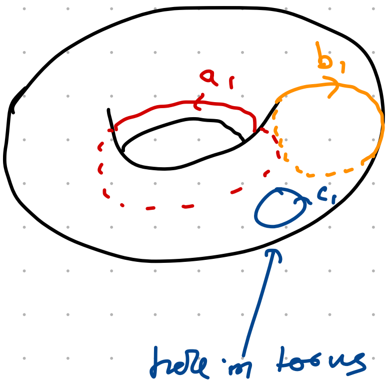
Surface of Genus 2

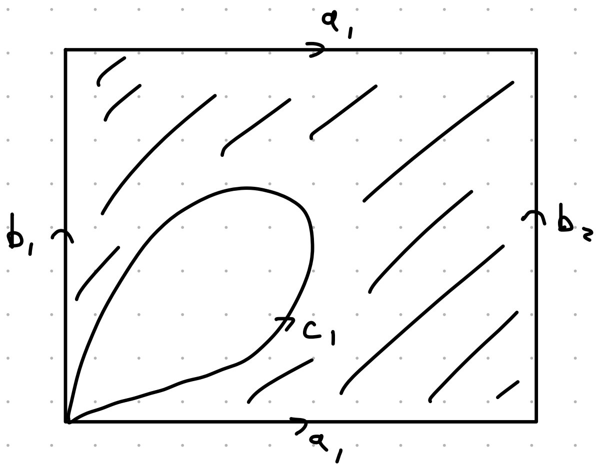


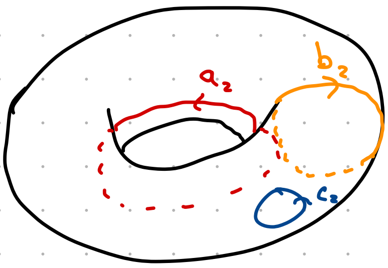


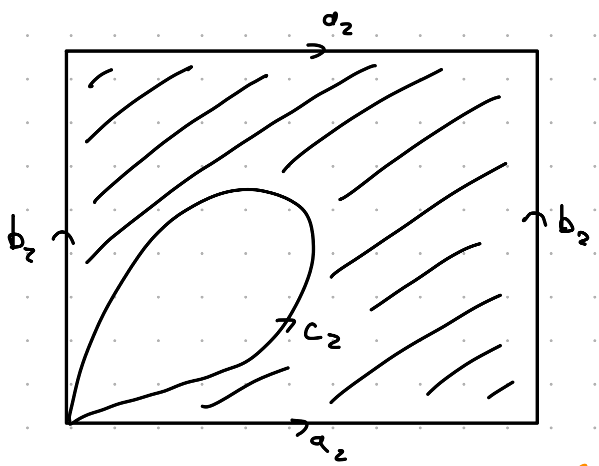
$= M_2$ [genus]

Generators a_1, b_2 horizontal & vertical

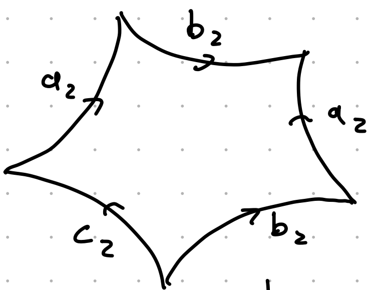


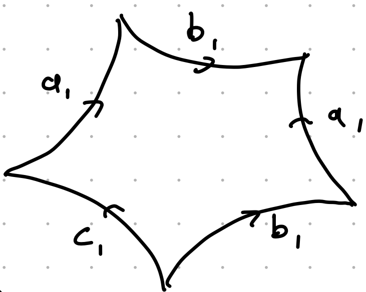




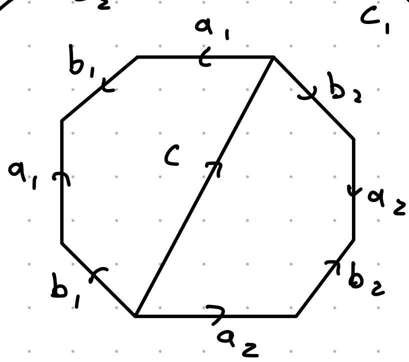


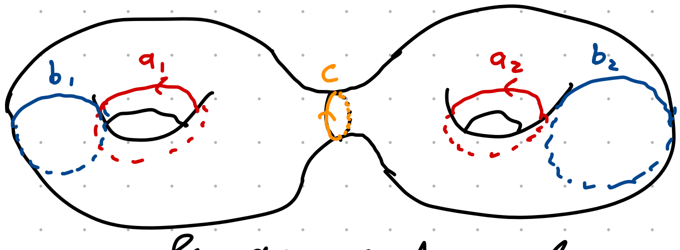
or





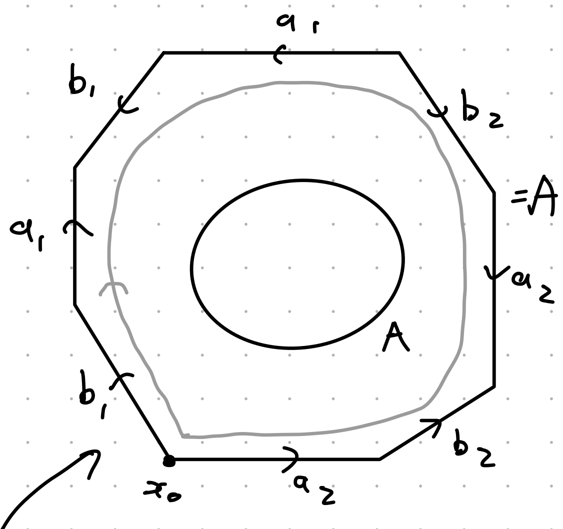
c is used to connect so can ignore it. If you include, you have two dists, want just one!

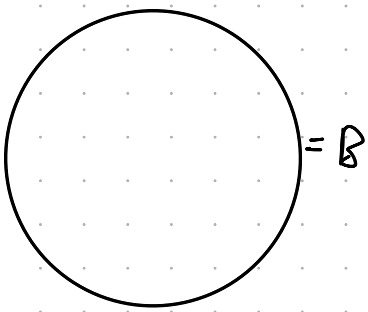


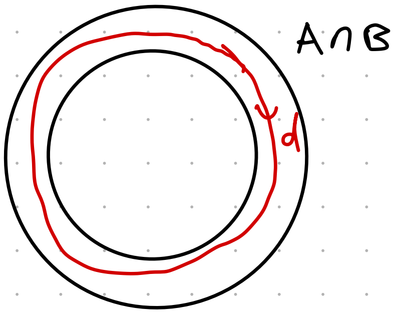


& any combo works...

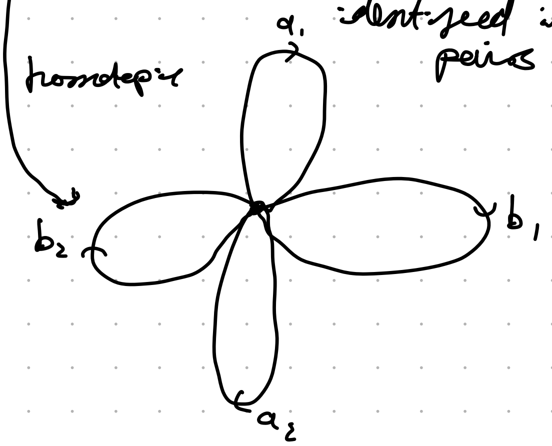
Calc fundamental group using same tactic as before







$\pi_1(A)$ = free group on 4 generators identified in pairs



$$\begin{array}{ccccc} \pi_1(A) & & \pi_1(A \cap B) & & \pi_1(B) \\ b_1 a_1 b_1^{-1} a_2^{-1} b_2 a_2 b_2^{-1} b_1^{-1} & \longleftarrow & d & \longmapsto & \{1\} \end{array}$$

$$\therefore \quad \pi_1(M_2) = \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) *_{\langle d \rangle} \{1\}$$

$$= \langle a_1, a_2, b_1, b_2 \mid \underbrace{\underbrace{b_1 a_1 b_1^{-1} a_1^{-1}}_{[a_1, b_1]} \cdot \underbrace{b_2 a_2 b_2^{-1} b_2^{-1}}_{[a_2, b_2]}}_{\text{comutators}} = 1 \rangle$$

$[a_1, b_1] \cdot [a_2, b_2] = 1$

<u>Thm:</u> If the free group on n generators is isomorphic to the free group on m generators, then $n = m$.

<u>Proof:</u> Consider $F(a_1, \dots, a_n)$ Evaluate $\mathrm{Hom}(F(a_1, \dots, a_n), \mathbb{Z}/2\mathbb{Z})$

cardinality of this set of homomorphisms is 2^n.

$$\begin{tikzcd} F(a_1, \dots, a_n) \arrow[drr, "\hat{f}"] & & \\ a_1, \dots, a_n \arrow[u, hook] \arrow[rr, "f"] & & \{0, 1\} \end{tikzcd}$$

Thm: For any group G, $\exists$ space X (CW complex) built by attaching disks to a graph s.t.

$$\pi_1(X) = G$$

Pf: Every group gets a representation. Use that.